Exploring Complex Analysis [1]
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Michael Brilleslyper

Michael Dorff

Jane McDougall

James Rolf

Beth Schaubroeck

Rich Stankewitz

Kenneth Stephenson

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United States Air Force Academy
E-mail address: Mike.Brilleslyper@usafa.edu

Brigham Young University
E-mail address: mdorff@math.byu.edu

Colorado College
E-mail address: JMcDougall@ColoradoCollege.edu

United States Air Force Academy
E-mail address: rolf.jim@gmail.com

United States Air Force Academy
E-mail address: Beth.Schaubroeck@usafa.edu

Ball State University
E-mail address: rstankewitz@bsu.edu

University of Tennessee, Knoxville
E-mail address: kens@math.utk.edu

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Introduction

This book is written for undergraduate students who have studied some complex analysis and want to explore some research topics in the field. We envision this book could be used as:

- a supplement for a standard undergraduate complex analysis course allowing students as a group or as individuals to explore a research topic;
- a guide for undergraduate research projects for an advanced student or a small group of students; or
- a resource for senior capstone courses.

The nature of this book is quite different from other mathematics texts. This book focuses on discovery, self-driven investigation, and creative problem posing. Some of the ideas are part of the standard fare learned in a course which focuses solely on the topic, while others may be new to the field. We want to inspire the reader to investigate, explore, form conjectures, and pursue mathematical ideas. Students are taken on a guided tour of the topics and are given many opportunities to stray from the text to pursue their own investigations.

Interlaced in the reading for each chapter are exercises, explorations using computer applets, and projects. These activities are an essential part of the students' learning of the topic. For this reason, most of these activities end with the phrase ***Try it out!*** to remind the student that the activity needs to be done before going on with the reading. The types of activities are as follows:

- **Examples** - The reader should be sure that she/he can follow the arguments and provide small details when needed.
- **Exercises** - These have a well-defined goal and should be done by the reader before going on to the next paragraph in the text. Skipping these would result in the reader missing an important skill or idea that will be fundamental to her/his understanding.
- **Explorations** - These also should be done before going on in the text. Generally, these do not have a well-defined problem that you are trying to "solve". Some may include undirected investigating or "playing" with applets. There is no specific outcome expected, but much will be gained by this activity. Such activities are at the heart of what this book is about since it is getting the student to explore on her/his own and see what she/he can come up with.
- **Small Projects** - These are more involved activities than those listed above and are optional. It may take up to a few weeks to complete a Small Project.
- **Large Projects** - These are similar to a Small Project, but on a larger scale. A Large Project could be a semester long project, a capstone project, or an honors thesis.
- **Additional Exercises** - Additional exercises may include any of the previous activities. However, these appear at the end of the chapter and are optional.

While working on these activities, it is a good habit to consider such questions as "Why was this problem posed?" "Why is it interesting?" "What if I changed the problem slightly, does it make it easier? harder? impossible?" "What does such a problem say about the general theory?" Thinking about such questions is what it means to do mathematics research and investige the unknown. Pausing to constantly ask new questions, investigate these questions, and mathematically describe these questions can make for very slow reading. The reader should measure progress, not by the number of pages read, but by the amount of independent thought given to the material. If the student reads just a few pages of a chapter and then becomes motivated to work on a problem or set of problems she/he devised on their own, the authors of this book would be delighted.

> "It is not so very important for a person to learn facts. For that he does not really need a college. He can learn them from books. The value of an education in a liberal arts college is not learning of many facts but the training of the mind to think something that cannot be learned from textbooks."–Albert Einstein (in Einstein: His Life and Times by Frank)

This book contains six research topics. Each topic is presented in a self-contained chapter that contains necessary background material, presentation of new material, exercises, explorations, and problems suitable for student projects, and several computer applets that allow the student to explore the topic. Also, each topic is a fairly recent area of research, and there are a lot of new questions to investigate. Here is a brief description of each of the chapters in this book:

(1) Complex Dynamics: This chapter investigates chaos and fractals as they relate to dynamical systems which come from iterating complex valued functions, i.e., given an initial value z_0 we consider the values $z_1 = f(z_0), z_2 = f(z_1) = f(f(z_0))$, $z_3 = f(z_2) = f(f(f(z_0))), \ldots$, and ask what kind of behavior we can have in this sequence z_n. Iteration in this sense arises in Newton's method for approximating roots of complicated functions, and so our chapter begins by asking such questions as: Which initial values will "work" for Newton's method (i.e., converge to a root)? If I change my initial value z_0 slightly, will I get similar or drastically different behavior? Often these questions are pursued computationally, visually, and experimentally with the aid of computer applets. We then extend our discussion by considering the iteration of any complex analytic map, which leads to a pursuit of the mathematics behind the famous Mandelbrot set, and much more.

(2) Soap films, Differential Geometry, and Minimal Surfaces: Minimal surfaces in $\mathbb{R}^3$ are beautiful geometric objects that minimize surface area locally. Visually, they can be thought of as saddle surfaces – at each point, the surface bends upward in one direction in the same amount as it bends downward in its perpendicular direction. Minimal surfaces are related to soap films that result when a wire frame is dipped in soap solution. In this chapter, we present the necessary background from differential geometry, a field of mathematics in which the ideas and techniques of calculus are applied to geometric shapes, to give an introduction to minimal surfaces. Then we use ideas from complex analysis to present a nice way to describe minimal surfaces and to relate the geometry of the surface with this description. This allows us to begin investigating some of the interesting properties that can be studied with the help of the applets.

(3) Applications to Flow Problems: Two dimensional vector fields are used to model and study a wide range of phenomena. Of particular interest are vector fields that are both irrotational and incompressible. Such fields can be used to model the velocity of an ideal fluid flowing in a region or the electric field in a region free of charges. Modeling two dimensional fluid flow is a standard application of the theory of conformal mappings in complex variables. This chapter takes a geometric and visual approach to explore this standard body of work and then extends it to several more applications. Fields of interest typically include various sources or sinks that generate or remove fluid from the flow. Throughout the chapter, examples, theory, and exercises are used to develop methods that allow fields to be modeled that are generated by all types of sources and sinks in a variety of regions. Also, we have provided the applet *FlowTool* that readily displays the streamlines for a field with various sources and sinks. The applet permits real-time dynamical experimentation with the field. Students with an interest in using technology to visualize mathematical objects will find many opportunities to explore their ideas, though these are not explicit exercises.

(4) Anamorphosis, Mapping Problems, and Harmonic Univalent Functions: Complex-valued analytic functions have many very nice properties that are not necessarily possessed by differentiable real-valued functions. For example, if you can differentiate such a complex-valued function one time, then you can differentiate it infinitely many times. In addition, complex-valued analytic functions can always be represented as a Taylor series, and they are conformal (that is, they preserve angles) at points where $f' \neq 0$. Why does an analytic function have these properties? If $f = u + iv$ is an analytic function, then its real part $u(x, y)$ and its imaginary part $v(x, y)$ each satisfy Laplace's equation and thus are both harmonic. Also, u and v satisfy the Cauchy-Riemann equations and are therefore harmonic conjugates of each other. In this chapter we discuss some ideas and problems related to a collection of univalent (i.e., one-to-one) complex-valued functions $f = u + iv$, where u and v satisfy Laplace's equation but not necessarily the Cauchy-Riemann equations. This collection of functions are known as *harmonic univalent functions* or mappings, and contain the collection of analytic univalent functions as a subset.

(5) Mappings to Polygonal Domains: A rich source of problems in analysis is determining when, and how, one can create a univalent (one-to-one) function from one region onto another. In this chapter, we consider the problem of mapping the unit disk onto a polygonal domain by two different classes of functions. First for analytic functions we give an overview of the well established Schwarz-Christoffel transformation. This method leads to some very rich mathematics, the study of special functions, so we give a brief primer of a few special functions. We then diverge from analytic function theory and consider the Poisson Integral Formula to find harmonic functions that will serve as mapping functions onto polygonal domains. Proving that these harmonic functions are univalent requires us to explore some less known theory of harmonic functions and some relatively new techniques.

(6) Circle Packing: *Circle packings* are configurations of circles with prescribed patterns of tangency. They exist in quite amazing and often visually stunning variety, but what are they doing in a book on complex analysis? Well, the fact is that complex analysis is at its heart a *geometric* topic. The reader will see this in the global geometry on display throughout Chapters 1–5, but the foundation lies down at the local level where, as the saying goes, "analytic functions map infinitesimal circles to infinitesimal circles." In Chapter 6 this geometry will come to life in the theory of *discrete analytic functions* based on circle packing. Using the Java application *CirclePack*, we will create, manipulate, and display maps between circle packings which are the discrete analogues of familiar functions between plane domains, including some of those encountered earlier in the book. Direct access to the underlying geometry gives new insight into fundamental topics like harmonic measure, extremal length, and branching. Moreover, we will see that our discrete functions not only mimic their classical counterparts, but actually converge to them under refinement. In short, Chapter 6 is about *quantum* complex analysis.

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Michael Dorff

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