CHAPTER 5

Mappings to Polygonal Domains

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5.1. Introduction

A rich source of problems in analysis is determining when, and how, one can create a one-to-one function of a particular type from one region onto another. In this chapter, we consider the problem of mapping the unit disk $\mathbb{D}$ onto a polygonal domain by two different classes of functions. First for analytic functions we give an overview and examples of the well known Schwarz-Christoffel transformation. We then diverge from analytic function theory and consider the Poisson integral formula to find harmonic functions that will serve as mapping functions onto polygonal domains. Proving that these harmonic functions are univalent requires us to explore some less known theory of harmonic functions and some relatively new techniques.

Because of the Riemann Mapping Theorem, we can simplify our mapping problem for *either* class of function to asking when we can map the unit disk $\mathbb{D} = \{z : |z| < 1\}$ univalently onto a target region. This is because if we want to map one domain (other than the entire set of complex numbers) onto another, we can first map it to $\mathbb{D}$ by an analytic function, and subsequently apply an analytic or harmonic mapping from $\mathbb{D}$ to the other domain (recall that the composition of a harmonic function with an analytic function is harmonic).

We begin in Section 1.2 with the Schwarz-Christoffel formula to find univalent analytic maps onto polygonal domains, and so set the stage for the corresponding problem for harmonic functions with the Poisson integral formula in Section 1.3. Perhaps because of their importance in applications, many first books on complex analysis introduce Schwarz-Christoffel mappings through examples, without emphasis on subtleties of the deeper theory. Our approach here will be the same, and the examples we include are chosen to bring together ideas found elsewhere in this book, such as the shearing technique from Chapter 4 and the construction of minimal surfaces from Chapter 5.

We also include an example of a Schwarz-Christoffel map onto a regular star, a polygonal domain that is highly symmetric but also non-convex. The problem of using the Poisson integral formula to construct a univalent harmonic function onto a non-convex domain is not at all well understood. In Section 1.3, after developing the theory for convex domains, we explore the example of finding harmonic maps onto regular star domains in detail, and lead the student to further investigation.

Terminology and technology: We use the term "univalent" for one-to-one, and take domain to mean open connected set in the complex plane. The applets used in this chapter are:

(1) *ComplexTool* - used to plot the image of domains in $\mathbb{C}$ under complex-valued functions.
(2) *PolyTool* - used to visualize the harmonic function that is the extension of a particular kind of boundary correspondence. The user of this applet can dynamically change the boundary correspondence and watch the harmonic function change.
(3) *StarTool* - used to examine the functions that map the unit disk $\mathbb{D}$ onto an n-pointed star. The user can modify the shape of the star (by changing n and r) and the boundary correspondence (by changing p).

5.2. Schwarz-Christoffel Maps

In this section we consider conformal maps from the unit disk and the upper half-plane onto various simply connected polygonal domains. By the Riemann Mapping Theorem, we can map the unit disk conformally onto any simply connected domain that is a proper subset of the complex numbers, with a mapping function that is essentially unique.

While the Riemann Mapping Theorem tells us that we **can** find a univalent analytic function to map $\mathbb{D}$ onto our domain, finding an actual Riemann mapping function is no easy task. Even for a simple domain such as a square, the mapping function from the disk cannot be expressed in terms of elementary functions. One situation however in which this problem *is* relatively simple is in mapping a region bounded by a circle or line in the complex plane to another such region, using fractional linear transformations. For this problem, we only need to pick three points on the bounding line or circle in the domain, and map them (in order) to three arbitrarily chosen points on the boundary of the target region (see for example [**10**], [**15**], [**19**], or [**20**]). This selection of three pairs of points determines the fractional linear transformation completely, and works for example, in finding a conformal map from $\mathbb{D}$ onto any planar region bounded by a line or circle.

EXERCISE 5.1. Show that the fractional linear transformation $z' = \phi(z) = i\frac{1+z}{1-z}$ maps the unit disk to the upper half-plane by finding the images of three boundary points. Then show that its inverse function $\phi^{-1}(z') = z = \frac{z'-i}{z'+i}$ maps the upper half-plane to the unit disk. ***Try it out!***

How can a mapping function be found in the case when the target region is more complicated? This question is relevant to solving the heat equation or the study of fluid flows, as discussed in Chapter 3.

The Schwarz-Christoffel transformation frequently enables us to find a function mapping onto a polygonal domain. In most texts the formula is presented as a mapping from the upper half-plane $\mathbb{H}$ onto the target polygon. We now develop this formula - for a more thorough treatment we refer the reader to [**13**], [**3**] and [**16**]. Suppose our target polygon has interior angles $\alpha_k\pi$ and exterior angles $\beta_k\pi$, where $\alpha_k + \beta_k = 1$ and $\alpha_k > 0$. The exterior angle measures the angle through which a bug, traversing the polygon in the counterclockwise direction, would turn at each vertex. This angle could be positive or negative, following the usual convention in mathematics that a counterclockwise rotation is positive while a clockwise rotation is negative. For example, in Figure 5.1 the angle marked by $\alpha_2\pi$ is $\frac{3\pi}{2}$, so $\alpha_2 = 3/2$. We can see from $\alpha_2 + \beta_2 = 1$ that $\beta_2 = -1/2$, which coincides with the description of $\beta_2\pi$ as a clockwise turn on the boundary. We can also obtain the exterior angle by extending one side of the polygon, and then seeing through what angle you would rotate that side to get to the next side of the polygon, as shown in the figure. For a simple closed polygon (that is, one with no self-intersections), it is always possible to describe the interior and exterior angles using coefficients α_k and β_k as described. As a final note about terminology, we describe a vertex such as the one described with β_1 in Figure 5.1 as a **convex corner** and the one described with β_2 as a **non-convex** corner.

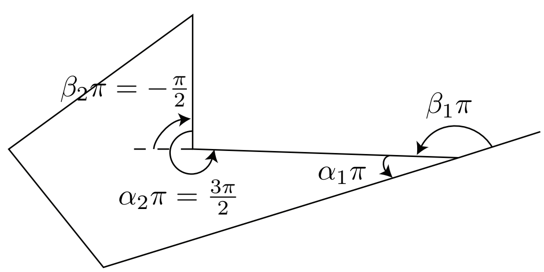

FIGURE 5.1. A sample polygon with both a convex and a non-convex corner

The Schwarz-Christoffel formula for the half-plane $\mathbb{H}$ to the polygon with exterior angles described by coefficients β_k as above is

$$f(z) = A_1 \int_0^z \frac{1}{(w-x_1)^{\beta_1}(w-x_2)^{\beta_2}\cdots(w-x_n)^{\beta_n}}dw + A_2, \quad z \in \mathbb{H}. \tag{67}$$

The real values x_i are preimages of the n vertices of the polygon, which we will refer to from now on as **prevertices**. Different choices of the constants A_1 and A_2 rotate, scale and/or translate the target n-gon.

In Equation 67, we use w as the variable of integration, and the limits of integration are chosen to make the definite integral into a function of z. The (abitrary) choice of 0 as a fixed point might have to be altered if it corresponds to a point of discontinuity of the integrand.

EXERCISE 5.2. You may be familiar with the sine and arcsine functions on the complex plane. Verify that the Schwarz-Christoffel mapping of $\mathbb{H}$ onto the infinite half strip described by $|\operatorname{Re}(z)| < \frac{\pi}{2}$ and $\operatorname{Im}(z) > 0$ is given by the arcsine function. Use the prevertices $x_1 = -1$, $x_2 = 1$ in formula 67. ***Try it out!***

We can observe that the angles at the vertices are represented in the formula, but nowhere do we see an accomodation for the side-lengths of the target polygon. In fact the side-lengths are influenced by the choice of prevertices x_i, but in a nonlinear and non-obvious way.

In the following example, we will apply the Schwarz-Christoffel formula to map the upper half-plane onto a rectangle. We will make a somewhat arbitrary choice of prevertices, and then evaluate the resulting integral to determine the target rectangle. In computing the integral, we come across a first example of a special function, and find that we need to learn some of the basics of elliptic integrals. We will find that just as the prevertices were arbitrarily chosen, so are the sidelengths of our target rectangle.

EXAMPLE 5.3. In this example, we map the upper half-plane $\mathbb{H}$ onto a rectangle. We will choose the prevertices $x_1 = -3, x_2 = -1, x_3 = 1$, and $x_4 = 3$. Since our target image is a rectangle, all of the exterior angles are $\pi/2$, so each $\beta_i = 1/2$. Using equation 67, we find that

$$f(z) = A_1 \int_0^z \frac{1}{(w-1)^{1/2}(w-3)^{1/2}(w+1)^{1/2}(w+3)^{1/2}} dw + A_2, \quad z \in \mathbb{H}.$$

The constant A_1 allows us to scale and rotate the image of $\mathbb{H}$, and A_2 allows for a translation. By chosing $A_1 = 1$ and $A_2 = 0$ we simplify to

$$f(z) = \int_0^z \frac{1}{\sqrt{(w^2-1)(w^2-9)}} dw. \tag{68}$$

This choice of constants does not affect the **aspect ratio** (ratio of adjacent sides) of the resulting rectangle. However the integral cannot be evaluated using techniques in standard calculus texts. Instead it is a special function known as an elliptic integral (of the first kind, with parameter $k = \frac{1}{3}$).

DEFINITION 5.4. An elliptic integral of the first kind is an integral of the form

$$F(\phi, k) = \int_0^{\sin\phi} \frac{1}{\sqrt{(1-w^2)(1-k^2w^2)}} dw.$$

An alternate form is $F(\phi, k) = \int_0^{\phi} \frac{1}{\sqrt{1-k^2\sin^2\theta}} d\theta$.

The two integrals in Definition 5.4 are identical after the change of variables $w = \sin\theta$, $dw = \cos\theta d\theta = \sqrt{1-w^2}d\theta$ which connects them. (Technology note: The computer algebra system *Mathematica* uses the alternate form, representing the integral by EllipticF$[\phi, m]$, where $m = k^2$.)

EXERCISE 5.5. Carry out the change of variables $w = \sin\theta$, $dw = \cos\theta d\theta = \sqrt{1-w^2}d\theta$ to show that

$$F(\phi, k) = \int_0^{\sin\phi} \frac{1}{\sqrt{(1-w^2)(1-k^2w^2)}}dw = \int_0^{\phi} \frac{1}{\sqrt{1-k^2\sin^2\theta}}d\theta.$$

Try it out!

Returning to our integral from Equation 68, we now work on rewriting it so that we may recognize it as an elliptic integral of the first kind.

$$\begin{aligned}
f(z) &= \int_0^z \frac{1}{\sqrt{(w^2-1)(w^2-9)}}dw \\
&= \int_0^z \frac{1}{\sqrt{(w^2-1)\left(w^2-\frac{1}{1/9}\right)}}dw \\
&= \int_0^z \frac{1}{\sqrt{\frac{1}{1/9}}}\frac{1}{\sqrt{(w^2-1)\left(\frac{1}{9}w^2-1\right)}}dw \\
&= \frac{1}{3}\int_0^z \frac{1}{\sqrt{(w^2-1)\left(\frac{1}{9}w^2-1\right)}}dw \\
&= \frac{1}{3}\int_0^z \frac{1}{\sqrt{(1-w^2)\left(1-\frac{1}{9}w^2\right)}}dw \\
&= \frac{1}{3}F(\arcsin z, \frac{1}{3}).
\end{aligned}$$

As we will see, our initial choice for the prevertices $(-3, -1, 1, 3)$ directly impacts the aspect ratio of the rectangle.

EXERCISE 5.6. Follow these steps to determine the aspect ratio of the rectangle that is the image of $\mathbb{H}$ under the function

$$f(z) = \frac{1}{3}F(\arcsin z, \frac{1}{3}).$$

(1) Explain why the integral

$$K_1 = \int_0^1 \frac{1}{\sqrt{(1-w^2)\left(1-\frac{1}{9}w^2\right)}}dw$$

must be a real number (hint: use geometry of the integrand). Conclude that

$$f(1) = \frac{1}{3}F\left(\arcsin 1, \frac{1}{3}\right) = \frac{1}{3}F\left(\pi/2, \frac{1}{3}\right) = K_1/3.$$

By symmetry, show that $f(-1) = -K_1/3$. Thus the length of one side of the rectangle is $2|K_1|/3$.

(2) Determine that since $f(3)$ is the next vertex of the target rectangle (moving counterclockwise), and $f(3) = f(1) + iK_2$ where K_2 is some real constant. Combine this with the fact that

$$f(3) = \frac{1}{3}F\left(\arcsin 3, \frac{1}{3}\right)$$

to show that $iK_2 = \frac{1}{3}\left(F(\arcsin 3, \frac{1}{3}) - F(\pi/2, \frac{1}{3})\right)$. The choice of sign for K_2 could be either positive or negative, depending on our choice of $\sqrt{-1}$. For consistency with our choice of angles, K_2 should be positive (note that *Mathematica* uses the "wrong" branch of the square root for this function on the real axis).

(3) Combine the findings above to determine that the aspect ratio of the rectangle is

$$\frac{2F\left(\pi/2, \frac{1}{3}\right)}{F\left(\arcsin 3, \frac{1}{3}\right) - F\left(\pi/2, \frac{1}{3}\right)} \approx 1.279...$$

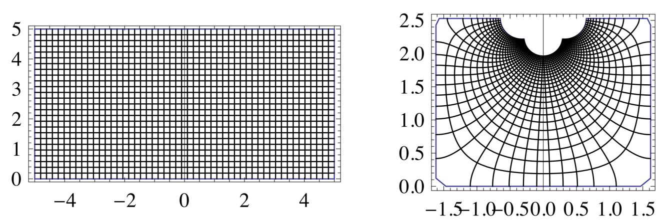

FIGURE 5.2. Portion of upper half-plane (left) and portion of target rectangle (right) to which it maps under the function of Exercise 5.3.

The mapping for Example 5.3 is illustrated in Figure 5.2. Only a portion of the upper half-plane and its image are shown. This explains why the target rectangle is incompletely filled in the upper central area. However we can see that the aspect ratio is at least approximately the same as the one we calculated.

SMALL PROJECT 5.7. Rework Example 5.3 for a more general situation. Use the prevertices $x_1 = -\lambda$, $x_2 = -1$, $x_3 = 1$, and $x_4 = \lambda$, where $\lambda > 1$. You can find an equation involving λ as a variable that, chosen correctly, would force the target rectangle to be a square. To make the target rectangle a square, you find that λ is part of an equation that cannot be explicitly solved, and must be approximated numerically. This is a standard problem in some introductory complex analysis books (see for instance example 22 of section 14, [**19**]).

One observation we can make based on this example is that while it is straightforward to write down a mapping function that has the correct angles, there is no simple way to prescribe the side-lengths. Also, there are only a few rare cases when our integral can be expressed in terms of elementary functions, and in general it is not easy to evaluate. In order to find and evaluate specific Schwarz-Christoffel mappings, it is usually helpful to use symmetry of the target polygon (and of the prevertices) to simplify the computations.

A further simplification to the Schwarz-Christoffel formula that is frequently employed is to set one of the prevertices to be ∞, which effectively removes one factor from the denominator of the Schwarz-Christoffel formula. An example where this simplification is helpful is in mapping onto a triangle.

We note here that there is no guarantee that the Schwarz-Christoffel formula will result in a univalent function (see [**9**]). The only thing we can say for sure is that a map from the upper half plane to a simply connected polygonal domain that is conformal MUST take the form of equation 67 for some choice of constants and prevertices (for more detail, see [**3**]).

In Exercise 5.6, we made use of symmetry by choosing the prevertices to be ± 1 and ± 3. This symmetry simplified our calculations of the rectangle's vertices. However we unable to easily find a mapping onto a square. In mapping onto a square, or onto any regular polygon, it makes sense to adapt the Schwarz-Christoffel formula to map the unit disk to the polygon, using symmetrically placed points on the unit circle in place of the x_i in our existing formula. This can be accomplished by precomposing our mapping function found with Equation 67 with a Möbius transformation from the unit disk to the upper half-plane discussed in Exercise 5.1.

In addition to the problem of prescibing the lengths of the sides of the target polygon, a further problem arises with this approach for target polygons more complicated than a rectangle. Typically we will produce an integral that cannot be evaluated, even with special functions. These two issues are nicely resolved if we instead obtain a Schwarz-Christoffel formula that maps the unit disk directly to the target polygon and with the preimages of the vertices falling on the unit circle. These points can be chosen, for example, to be roots of unity to give some symmetry in the integral. To obtain this formula we carry out a change of variables that maps the disk to the upper half-plane, using the map defined in Exercise 5.1. Interestingly, the transformed integral formula is of exactly the same form.

EXERCISE 5.8. Set $w = \phi(z) = i\frac{1+z}{1-z}$ which maps the disk in the z plane to the upper half w-plane (see Exercise 5.1). Show that the Schwarz-Christoffel formula retains the same form as equation 67. ***Try it out!***

The Schwarz-Christoffel map that we will use on the unit disk is then

$$f(z) = C_1 \int_0^z \frac{1}{(w-\zeta_1)^{\beta_1}(w-\zeta_2)^{\beta_2}\cdots(w-\zeta_n)^{\beta_n}} dw + C_2, \ z \in \mathbb{D},$$

where $\beta_i \pi$ is the exterior angle of the ith vertex of the target polygon, and the preimages ζ_i of the vertices are on the unit circle. Here we use ζ_i instead of x_i to emphasize that the prevertices are not on the real axis. As with Equation 67, the complex constants C_1 and C_2 with $C_1 \neq 0$ rotate, resize and translate the polygon.

With our original Schwarz-Christoffel formula from the upper half plane, it is not at all obvious how we could obtain a map onto a regular polygon. However, we can exploit the symmetries of the roots of unity by choosing ζ_i to be the symmetrically placed nth roots of unity corresponding to symmetrically placed vertices in the target polygon. Consequently, all side lengths of the target polygon will be equal.

Example 5.9. We obtain the Schwarz-Christoffel map onto a regular n-gon. The exterior angles of a regular n-gon are $2\pi/n$, so $\beta_i = 2/n$.

$$\int_0^z \frac{1}{(w-\zeta_1)^{\beta_1}(w-\zeta_2)^{\beta_2}\cdots(w-\zeta_n)^{\beta_n}}dw = \int_0^z \frac{1}{[(w-\zeta_1)(w-\zeta_2)\cdots(w-\zeta_n)]^{2/n}}dw$$

Suppose that the ζ_i are the nth roots of unity. Now we can use the fact that

$$\prod_{i=1}^{n}(w-\zeta_i) = w^n - 1$$

to simplify to $\int_0^z \frac{1}{(w^n-1)^{2/n}}dw$. By factoring out $(-1)^{2/n}$ we can adjust the multiplicative constant and chose our mapping function

$$f(z) = (-1)^{-2/n}\int_0^z \frac{1}{(w^n-1)^{2/n}}dw = \int_0^z \frac{1}{(1-w^n)^{2/n}}dw. \tag{69}$$

Here f has been defined from the Schwarz-Christoffel formula with choices of constant $C_1 = (-1)^{-2/n}$ (which rotates the figure by $4\pi/n$ radians) and $C_2 = 0$. This last formula cannot be evaluated using the usual methods from calculus, but can be readily evaluated using hypergeometric functions.

5.2.1. Basic Facts about Hypergeometric Functions.

The integral in the last example cannot be expressed in terms of elementary functions, but can be easily evaluated and plotted using a computer algebra system by using some basic facts about some special power series known as hypergeometric functions. Hypergeometric functions, besides their many other applications, can be used to evaluate the integrals obtained above. A geometric series is a power series in which ratios of successive terms are constant. Generalizing this, for a hypergeometric series ratios of successive terms are rational functions of the index rather than just constants. Here we will make use of the most widely utilized hypergeometric functions–the so-called "two F ones," where the rational function has numerator and denominator of the second order.

Definition 5.10. The hypergeometric function ${}_2F_1(a,b;c;z)$ is the power series

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},$$

where a, b and $c \in \mathbb{C}$ and

$$(x)_n = x(x+1)\cdots(x+n-1)$$

is the **shifted factorial**, or **Pochhammer symbol**.

EXERCISE 5.11. Use simple algebra to check that $(x)_{n+1} / (x)_n = x + n$. ***Try it out!***

If we compute the ratio of two successive terms in the geometric series $\sum_{n=0}^{\infty} r^n z^n$ we obtain simply the ratio r times z. In the next exercise we carry out the same computation for a hypergeometric series.

EXERCISE 5.12. Show that the ratio of two successive terms in the series ${}_2F_1(a, b; c; z)$ is

$$\frac{(a+n)(b+n)z}{(c+n)(n+1)}.$$

Try it out!

The formula obtained in this last exercise motivates the use of the term "hypergeometric." Whereas for a geometric series the ratio of successive terms is a single constant times z, for a hypergeometric function this ratio is a rational function of n, multiplied by z.

EXERCISE 5.13. Apply the ratio test to show that we get convergence of the hypergeometric function ${}_2F_1(a, b; c; z)$ on compact subsets of the unit disk. ***Try it out!***

Any function which is useful or widely applicable typically earns the status of "special function." A number of well known special functions can be written as hypergeometric series. For example

$$\begin{aligned} \log \frac{1+z}{1-z} &= 2z\,{}_2F_1\left(1/2, 1; 3/2; z^2\right), \\ (1-z)^{-a} &= {}_2F_1(a, b; b; z), \text{ and} \\ \arcsin z &= z\,{}_2F_1\left(1/2, 1/2; 1; z^2\right). \end{aligned}$$

The functions $\sin(z)$ and $\cos(z)$ themselves can each be obtained as limiting cases of a "two F one" series.

Now we consider another example that involves a ${}_2F_1$ hypergeometric series, $z\,{}_2F_1(\frac{1}{2}, \frac{1}{4}; \frac{5}{4}; z^4)$. We will see shortly that this function is a Schwarz-Christoffel transformation that maps the unit disk onto a square.

EXERCISE 5.14. Use Definition 5.10 to find the first several terms in the series of $z\,{}_2F_1(\frac{1}{2}, \frac{1}{4}; \frac{5}{4}; z^4)$. The following table gives the first several values of the necessary Pochhammer symbols. If you graph your result using *ComplexTool*, you should get a picture similar to Figure 5.3.

n	$(1/2)_n$	$(1/4)_n$	$(5/4)_n$	Coefficient of z^{4n+1}
0	1	1	1	1
1	1/2	1/4	5/4	1/10
2	3/4	5/16	45/16	1/24
3	15/8	45/64	585/64	5/208
4	105/16	585/256	9945/256	35/2176
5	945/32	9945/1024	208845/1024	3/256

Try it out!

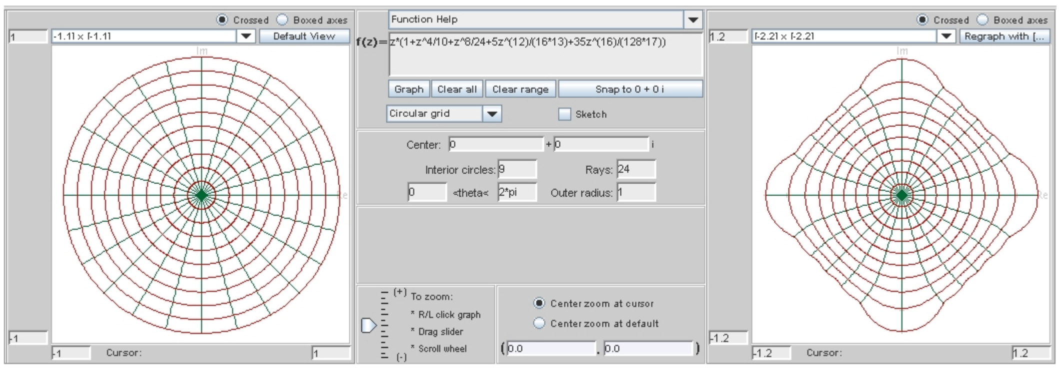

FIGURE 5.3. *ComplexTool* image of an approximation of the conformal map (using the first 5 terms)

The rate of convergences in this example is such that even with just a few non-zero terms of the series, we obtain a map whose image is approximately a square. We now see how to derive the formula for a Schwarz-Christoffel map onto the square.

DEFINITION 5.15 (Euler representation). The hypergeometric function ${}_2F_1(a,b;c;z)$ can be written in integral form as

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a} dt, $$

which is known as the *Euler representation* of the ${}_2F_1$ function.

The symbol Γ stands for the Gamma function, defined for z in the right half plane by

$$ \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt. $$

The function can be extended analytically to the whole plane except for the negative integers $-1, -2, -3, \ldots$

EXERCISE 5.16. For integer values of n, the Gamma function is related to the factorial by $\Gamma(n) = (n-1)!$. Prove this by directly evaluating $\Gamma(1) = 1$ and then showing that $\Gamma(z+1) = z\Gamma(z)$ (hint: use integration by parts). ***Try it out!***

EXERCISE 5.17. In this exercise, you will show that Definitions 5.10 and 5.15 are equivalent.

(1) Expand the factor $(1-tz)^{-a}$ with the binomial theorem as a power series to obtain

$$(1-tz)^{-a}=\sum_{n=0}^{\infty}\frac{(a)_n}{n!}t^n z^n.$$

(2) Use the formula (see [**14**] Theorem 7, p. 19 for a proof)

$$\Gamma(p)\,\Gamma(q)=\Gamma(p+q)\cdot\int_0^1 t^{p-1}(1-t)^{q-1}\,dt$$

(where p and q have positive real parts) to show

$$\int_0^1 t^{n+b-1}(1-t)^{c-b-1}dt=\frac{\Gamma(b+n)\,\Gamma(c-b)}{\Gamma(c+n)}.$$

The integral on the right is also an important special function known as the **beta function** in the variables p and q.

(3) Show

$$\frac{(b)_n}{(c)_n}=\frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\frac{\Gamma(b+n)\,\Gamma(c-b)}{\Gamma(c+n)}.$$

(4) Put the previous facts together to obtain the required formula. Substitute the power series for the denominator term, and then interchange summation and integral, to show that

$$\int_0^1\frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a}dt=\frac{\Gamma(b)\,\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1(a,b;c;z).$$

Try it out!

It takes a little work to establish the relationship of the beta function with the Gamma function used in part (2). For an excellent exposition of this fact and an introduction to special functions in general, see [**14**].

EXAMPLE 5.18. For a square (a regular 4-gon), the Schwarz Christoffel map from the unit disk is given by $z\,{}_2F_1(\frac{1}{2},\frac{1}{4};\frac{5}{4};z^4)$. These numbers probably seem to have been essentially "pulled from a hat," but when we apply the Euler integral representation we will see that these are the numbers we need (after a transformation) to evaluate the given integral. To see this, with $n=4$ in the integral representation of equation 69 we obtain

$$\int_0^z\frac{1}{\sqrt{1-w^4}}dw.$$

Let $a = 1/4$, $b = 1/2$, and $c = 5/4$. We use the Euler integral representation for ${}_2F_1(a, b; c; z)$ to evaluate

$$\begin{aligned} {}_2F_1\left(1/2, 1/4; 5/4; z^4\right) &= \frac{\Gamma(5/4)}{\Gamma(1/4)\Gamma(1)} \int_0^1 \frac{t^{-3/4}(1-t)^0}{(1-tz^4)^{1/2}} dt \\ &= 1/4 \int_0^1 \frac{1}{t^{3/4}} \frac{1}{\sqrt{1-tz^4}} dt. \end{aligned}$$

Now change variables by letting $w^4 = tz^4$ (so $t = (w^4/z^4)$) and $4w^3 dw = z^4 dt$. Then

$$\begin{aligned} {}_2F_1\left(1/2, 1/4; 5/4; z^4\right) &= 1/4 \int_0^z \frac{z^3}{w^3} \frac{1}{\sqrt{1-w^4}} \frac{4w^3 dw}{z^4} \\ &= \frac{1}{z} \int_0^z \frac{1}{\sqrt{1-w^4}} dw. \end{aligned}$$

Thus

$$f(z) = \int_0^z \frac{1}{\sqrt{1-w^4}} dw = z\, {}_2F_1\left(1/2, 1/4; 5/4; z^4\right).$$

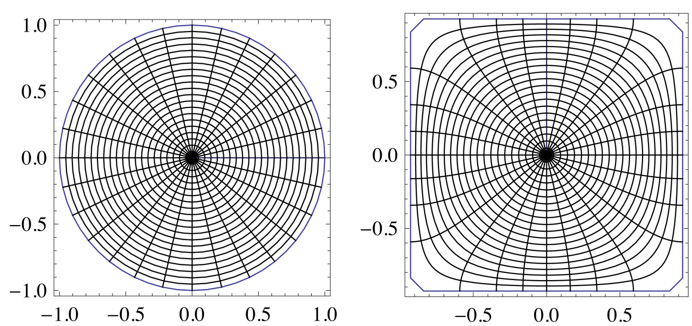

FIGURE 5.4. Unit disk (left) and target square (right) to which it maps under the function of Exercise 5.18.

Contrast the map in Figure 5.4, where the domain is the (bounded) unit disk, with the earlier map onto a rectangle in Figure 5.2. One advantage with the disk map is that we can see the entire mapping domain and the entire target square is filled. Note also the rotational and reflectional symmetry obtained by using nth roots of unity on the unit circle as prevertices.

EXERCISE 5.19. Show that the conformal map from the disk onto the regular n-gon is (up to rotations, translations and scalings) given by $z\, {}_2F_1\left(2/n, 1/n; (n+1)/n; z^n\right)$. ***Try it out!***

We describe another situation where this technique of integration can be useful, for readers who have worked through Chapters 2 or 4. In Chapter 4, Section 4.5 discusses the shear construction, and in Chapter 2, Section 2.6 the shear construction and its relationship to minimal surfaces is discussed.

Small Project 5.20. Define a non-convex 6-sided polygon P with $\beta_1 = \beta_4 = -1/3$ and $\beta_2 = \beta_3 = \beta_5 = \beta_6 = 2/3$. (Draw this polygon!) Find a representation for the Schwarz-Christoffel transformation that maps the unit disk $\mathbb{D}$ onto P, with the prevertices ζ_i being the 6th roots of unity. It works well if $\zeta_1 = 1$ and $\zeta_4 = -1$, with the other 6th roots of unity going in order counterclockwise around the circle. Verify that the analytic function $f(z) : P \to \mathbb{D}$ is given by

$$f(z) = z\,{}_2F_1\left(\frac{2}{3}, \frac{1}{6}; \frac{7}{6}; z^6\right) - \frac{z^3}{3}\,{}_2F_1\left(\frac{2}{3}, \frac{1}{2}; \frac{3}{2}; z^6\right).$$

Now, in the language of Chapter 4, let $h(z) - g(z)$ be the function given above as $f(z)$ and let the dilatation $\omega(z) = z^2$. Find the harmonic function $h(z) + \overline{g(z)}$. Verify that by using

$$h(z) = z\,{}_2F_1\left(\frac{2}{3}, \frac{1}{6}; \frac{7}{6}; z^6\right)$$

and

$$g(z) = \frac{z^3}{3}\,{}_2F_1\left(\frac{2}{3}, \frac{1}{2}; \frac{3}{2}; z^6\right),$$

it is indeed true that $\omega(z) = \frac{h'}{g'} = z^2$. Use a computer algebra system to create a picture of the image of $\mathbb{D}$ under the function $h(z) + \overline{g(z)}$.

Furthermore, if you have studied Chapter 2, you can find the minimal surface that lifts from this harmonic function. You should find that it is defined by

$$\begin{aligned}
x_1 &= \operatorname{Re}\left(z\,{}_2F_1\left(\frac{2}{3}, \frac{1}{6}; \frac{7}{6}; z^6\right) + \frac{z^3}{3}\,{}_2F_1\left(\frac{2}{3}, \frac{1}{2}; \frac{3}{2}; z^6\right)\right)\\
x_2 &= \operatorname{Im}\left(z\,{}_2F_1\left(\frac{2}{3}, \frac{1}{6}; \frac{7}{6}; z^6\right) - \frac{z^3}{3}\,{}_2F_1\left(\frac{2}{3}, \frac{1}{2}; \frac{3}{2}; z^6\right)\right)\\
x_3 &= \operatorname{Im}\left(z^2\,{}_2F_1\left(\frac{2}{3}, \frac{1}{3}; \frac{3}{2}; z^6\right)\right).
\end{aligned}$$

We now examine the conformal map onto a symmetric non-convex polygon in the shape of a star. We intend in the next section to find harmonic maps onto the same figure, and will find that while mappings onto polygons with convex corners are relatively easy to construct, there is little or no supporting theory when non-convex corners are involved.

Example 5.21. Suppose we want to map onto a (non-convex) m-pointed star, so there are $n = 2m$ vertices. The interior angles alternate between $\pi\alpha_1$ and $\pi\alpha_2$ where $\alpha_1 < 1 < \alpha_2$ (so a "sharp point" of the star occurs at α_1). Corresponding exterior angles then alternate between a positive value β_1 and a negative value β_2 (assuming we have a non-convex star).

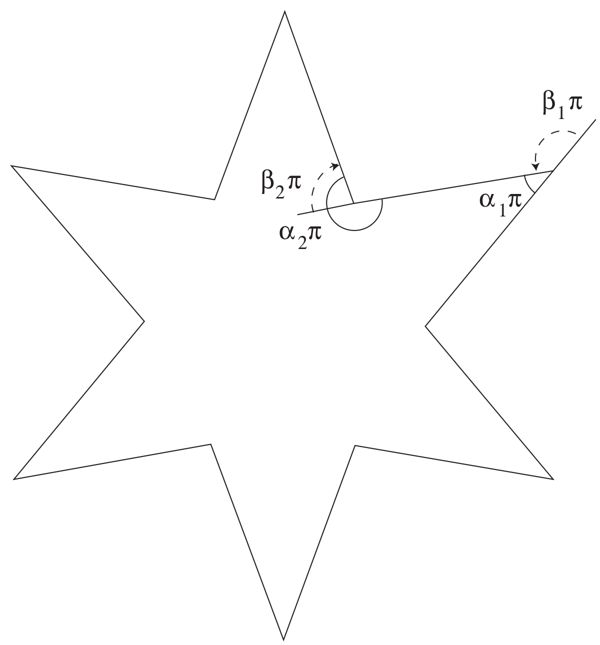

FIGURE 5.5. Interior and exterior angles of a symmetric star

Also, $\beta_1 + \beta_2$ must satisfy $m\,(\beta_1 + \beta_2) = 2$ so $\beta_1 + \beta_2 = 2/m = 4/n$. We use $\beta_{odd} = \beta_1 > 0$ and $\beta_{even} = -\beta_2 > 0$. Thus we have

$$\int_0^z \frac{\displaystyle\prod_{i \text{ even}} (w - \zeta_1)^{\beta_i}}{\displaystyle\prod_{i \text{ odd}} (w - \zeta_i)^{\beta_i}} dw.$$

Letting ζ_i be nth roots of unity,

$$\prod_{i \text{ even}}^{n} (z - \zeta_i) = z^m - 1 \text{ and } \prod_{i \text{ odd}}^{n} (z - \zeta_i) = z^m + 1,$$

so

$$\int_0^z \frac{(w^m - 1)^{-\beta_2}}{(w^m + 1)^{\beta_1}} dw,$$

where $\beta_1 + \beta_2 = 4/n$. Apart from constants chosen to expand or rotate the figure as necessary, we obtain the mapping function

$$f\,(z) = \int_0^z \frac{(1 - w^m)^{-\beta_2}}{(1 + w^m)^{\beta_1}} dw$$

from the disk onto the star shape in Figure 5.5.

EXAMPLE 5.22. The following appears as an exercise in [**13**] (Chapter V). Prove that the integral that maps the unit disk exactly onto a 5-pointed star with interior angles alternating at $\pi/5$ and $7\pi/5$ is given by

$$f(z) = \int_0^z \frac{(1-w^5)^{2/5}}{(1+w^5)^{4/5}} dw.$$

The corresponding exterior angles are $4\pi/5$ and $-2\pi/5$, so $\beta_1 = 4/5$ and $\beta_2 = -2/5$. Thus we have $n = 10$ and $m = 5$ and

$$\int_0^z \frac{(z^m-1)^{-\beta_2}}{(z^m+1)^{\beta_1}} dz = \int_0^z \frac{(z^5-1)^{2/5}}{(z^5+1)^{4/5}} dz$$

To compute this integral we must use the Appell F_1 function of two variables defined below.

DEFINITION 5.23. The Appell F_1 function is defined by

$$F_1(a; b_1, b_2; c; x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(a)_{n+m} (b_1)_m (b_2)_n}{m! n! (c)_{n+m}} x^m y^n,$$

In *Mathematica* one can use the command `AppelF1(a,b`$_1$`,b`$_2$`,c,x,y)`. This special function also has an integral form, just as the hypergeometric functions have the Euler representation. We do not include a derivation here but refer the interested reader to [**2**] (Chapter 9) for the integral formula

$$F_1(a; b_1, b_2; c; x, y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 u^{a-1} (1-u)^{c-a-1} (1-ux)^{-b_1} (1-uy)^{-b_2} du.$$

Working in reverse we find that

$$\begin{aligned} &F_1\left(1/5; 4/5, -2/5; 6/5; z^5, -z^5\right) \\ &= \frac{\Gamma(6/5)}{\Gamma(1/5)\Gamma(1)} \int_0^1 u^{-4/5} (1-u)^0 \left(1-uz^5\right)^{-4/5} \left(1+uz^5\right)^{2/5} du \\ &= 1/5 \int_0^1 \frac{(1-uz^5)^{2/5}}{u^{4/5} (1+uz^5)^{4/5}} du. \end{aligned}$$

To obtain our Schwarz-Christoffel formula we must now change variables, letting $w^5 = uz^5$ so $5w^4 dw = z^5 du$. Then

$$\begin{aligned} F_1\left(1/5; 4/5, -2/5; 6/5; z^5, -z^5\right) &= 1/5 \int_0^z \frac{z^4}{w^4} \frac{(1-w^5)^{2/5}}{(1+w^5)^{4/5}} \frac{5w^4 dw}{z^5} \\ &= \frac{1}{z} \int_0^z \frac{(1-w^5)^{2/5}}{(1+w^5)^{4/5}} dw \end{aligned}$$

Thus

$$f(z) = \int_0^z \frac{(1-w^5)^{2/5}}{(1+w^5)^{4/5}} dw = z\, F_1\left(1/5; 4/5, -2/5; 6/5; z^5, -z^5\right)$$

and the mapping function is shown in Figure 5.6.

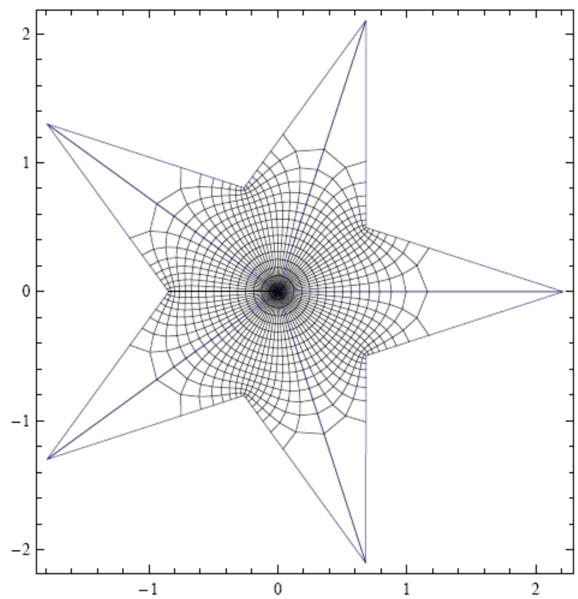

FIGURE 5.6. Image of the conformal map of the unit disk onto the 5-pointed star

EXERCISE 5.24. Show that the conformal map from the disk onto the m pointed star with exterior angle $\beta_1 > 0$, and $\beta_2 = 2/m - \beta_1$ (up to rotations, translations and scalings) is given by $z\, F_1\left(1/n; \beta_1, \beta_2, (n+1)/n; z^n, -z^n\right)$ where F_1 is the Appell F_1 function. ***Try it out!***

5.3. The Poisson Integral Formula

While the Schwarz-Christoffel formula gives analytic and thus angle-preserving (conformal) functions from $\mathbb{D}$ to any polygon, we can see that it often starts with an integral that requires advanced mathematics to evaluate. If our goal is not necessarily an analytic function, we could work with the Poisson integral formula. This does not give us an analytic function, but instead, a harmonic function from the unit disk to the target domain. We first recall the definition of a harmonic function.

DEFINITION 5.25. A real-valued function $u(x, y)$ is harmonic provided that

$$u_{xx} + u_{yy} = 0.$$

A complex-valued function $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ is harmonic if both u and v are harmonic.

The definition of a complex-valued harmonic function does not require that the functions u and v be harmonic conjugates, so while all analytic functions are harmonic, a complex-valued harmonic function is not necessarily analytic. In fact, the functions we work with for the rest of the chapter will not be analytic, and thus not conformal.

You may be familiar with the Poisson integral formula as a way of constructing a real-valued harmonic function that satisfies certain boundary conditions. For example, if the boundary conditions give the temperature of the boundary of a perfectly insulated plate, then the harmonic function gives the steady-state temperature of the interior of the plate. Another application is to find electrostatic potential given boundary conditions. A brief summary of that procedure is given here. For more detailed discussion, consult [**15**] or [**20**].

Theorem 5.26 (Poisson Integral Formula). Let the complex valued function $\hat{f}(e^{i\theta})$ be piecewise continuous and bounded for θ in $[0, 2\pi]$. Then the function $f(z)$ defined by

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2}\hat{f}(e^{it})dt \tag{70}$$

is the unique harmonic function in the unit disk that satisfies the boundary condition

$$\lim_{r\to 1} f(re^{i\theta}) = \hat{f}(e^{i\theta})$$

for all θ where $\hat{f}$ is continuous.

Here, we present the proof in the special case where the boundary function $\hat{f}(e^{i\theta})$ is the real part of a function that is analytic on a disk with radius larger than 1. This proof can be found in any standard complex analysis textbook, for example, [**10**] or [**15**]. The interested reader may find the full result in Chapter 6 of [**?**] and Chapter 8 of [**10**].

Proof. (Special Case) First observe that Cauchy's integral formula tells us that if we have a function $f(z)$ that is analytic inside and on the circle $|z| = R$, then, for $|z| < R$,

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta|=R} \frac{f(\zeta)}{\zeta - z} d\zeta. \tag{71}$$

Here we are using the Greek letter zeta (ζ) as the variable of integration in the integral. In the discussion that follows here, we will be thinking about evaluating the function $f(z)$ at some fixed value of z, so the variable under consideration is now ζ. We also observe (for reasons that will become obvious in a few sentences) that for fixed z, with $|z| < 1$, the function $\dfrac{f(\zeta)\,\overline{z}}{1-\zeta\,\overline{z}}$ is analytic in the variable ζ on and inside

$|\zeta| < 1$, since the denominator is non-zero. (Exercise for the reader: Think about why the denominator is non-zero.) Thus, by the Cauchy Integral Theorem, we know that

(72) $$\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{f(\zeta)\,\overline{z}}{1-\zeta\,\overline{z}} = 0.$$

Combining Equations 71 and 72, we see that

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{f(\zeta)}{\zeta - z}d\zeta + 0 \\ &= \frac{1}{2\pi i}\int_{|\zeta|=1}\left(\frac{f(\zeta)}{\zeta - z} + \frac{f(\zeta)\,\overline{z}}{1-\zeta\,\overline{z}}\right)d\zeta \\ &= \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{1-\zeta\,\overline{z} + \overline{z}(\zeta - z)}{(\zeta - z)(1-\zeta\,\overline{z})}f(\zeta)d\zeta \\ &= \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{1-|z|^2}{(\zeta - z)(1-\zeta\,\overline{z})}f(\zeta)d\zeta. \end{aligned}$$

Now we parameterize the circle $|\zeta| = 1$ by $\zeta(t) = e^{it}$, giving $d\zeta = ie^{it}dt$ and

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_0^{2\pi}\frac{1-|z|^2}{(e^{it}-z)(1-e^{it}\,\overline{z})}f(e^{it})ie^{it}dt \\ &= \frac{1-|z|^2}{2\pi}\int_0^{2\pi}\frac{1}{(e^{it}-z)e^{it}(e^{-it}-\,\overline{z})}f(e^{it})e^{it}dt \\ &= \frac{1-|z|^2}{2\pi}\int_0^{2\pi}\frac{f(e^{it})}{(e^{it}-z)(e^{-it}-\,\overline{z})}dt. \end{aligned}$$

Taking the real part of both sides of the equation gives us a harmonic function (since the real part of an analytic function is harmonic), and also yields Equation 70.

□

EXERCISE 5.27. Verify that the "Poisson kernel," $\dfrac{1-|z|^2}{|e^{it}-z|^2}$, can be rewritten as $\operatorname{Re}\left(\dfrac{e^{it}+z}{e^{it}-z}\right) = \operatorname{Re}\left(\dfrac{1+ze^{-it}}{1-ze^{-it}}\right)$. ***Try it out!***

The integral in Equation 70 is, in general, very difficult to integrate. However, if there is an arc on which the function $\hat{f}(e^{it})$ is constant, then the integration is easy to do.

EXERCISE 5.28. Verify that

(73)
$$\frac{1}{2\pi}\int_a^b K\operatorname{Re}\left(\frac{e^{it}+z}{e^{it}-z}\right)dt = K\frac{b-a}{2\pi}+\frac{K}{\pi}\arg\left(\frac{1-ze^{-ib}}{1-ze^{-ia}}\right) = \frac{K}{\pi}\left[\arg\left(\frac{e^{ib}-z}{e^{ia}-z}\right) - \frac{b-a}{2}\right].$$

Helpful hints: Swap the order of integration and taking the real part of a function. The algebraic identity $\frac{1+w}{1-w} = 1 + \frac{2w}{1-w}$ will be helpful.

Try it out!

The beauty of the last formulation of equation (73) is that it can be visualized geometrically. Consider the picture with e^{ia} and e^{ib} on the unit circle, and z somewhere inside the unit circle. Then the vector from z to e^{ia} is $e^{ia} - z$, and the vector from z to e^{ib} is $e^{ib} - z$, so that the angle between those two vectors is given by $\arg\left(\dfrac{e^{ib} - z}{e^{ia} - z}\right)$, as is shown if Figure 5.7.

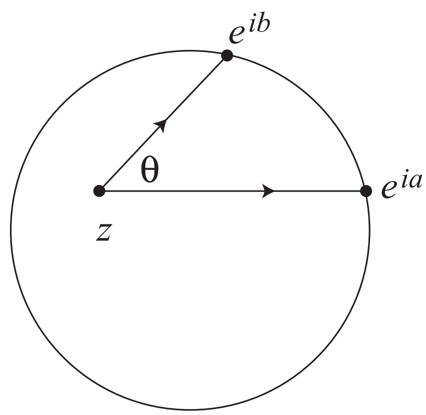

FIGURE 5.7. Geometric interpretation of $\theta = \arg\left(\dfrac{e^{ib} - z}{e^{ia} - z}\right)$.

EXAMPLE 5.29. Assume that the unit disk is a thin insulated plate, with a temperature along the boundary of 50 degrees for the top semicircle and 20 degrees along the bottom semicircle. From physics, we know that the function which describes the temperature within the unit disk must be a harmonic function. Use the results above to find that harmonic function.

Solution: Apply the formula given above to the situation where $a_1 = 0, b_1 = \pi, K_1 = 50$ and then add it to the result where $a_2 = \pi, b_2 = 2\pi, K_2 = 20$. The result is the function $f(z) = \frac{1}{2\pi}(70\pi + 60 \arg\left(\frac{1+z}{1-z}\right))$. (Notice that the 70 is $50 + 20$, and that $60 = 2(50 - 20)$.) When z ranges across the unit disk, the function $\frac{1+z}{1-z}$ covers the right half-plane (you can check this experimentally by graphing the function $\frac{1+z}{1-z}$ using *ComplexTool*), so the argument of it is between $-\pi/2$ and $\pi/2$. This gives function values for $f(z)$ between 20 and 50, which makes good sense. Another way of thinking of the solution is that it gives the average temperature $\pm$ half of the difference between the maximum and minimum temperatures.

EXERCISE 5.30. Referring to $f(z) = \frac{1}{2\pi}(70\pi + 60 \arg\left(\frac{1+z}{1-z}\right))$ in the solution to Example 5.29, find $f(0)$, $f(i/2)$ and $f(-i/2)$. Do your answers make sense?

Using the result of Exercise 5.28, we can see that computing the Poisson integral formula for a *piecewise constant* boundary is particularly simple. Many applications of the Poisson integral formula come from having the boundary correspondence remain constant on arcs of the unit circle.

Most introductory analysis books give the Poisson integral formula for real-valued $\hat{f}(e^{i\theta})$. It can also be applied to create a harmonic function for complex-valued $\hat{f}(e^{i\theta})$, but the univalence of the harmonic function is not at all apparent. Let's first explore what could happen if we try to use the Poisson integral formula with complex boundary values.

EXAMPLE 5.31. The simplest example of this is obtained by letting the first third of the unit circle (that is, the arc from 0 to $e^{i2\pi/3}$) map to 1, the next third to $e^{i2\pi/3}$ and the last third to $e^{i4\pi/3}$. Let's work through the details of this integration, working from equation (73). We compute

$$\begin{aligned}
f(z) &= \frac{1}{2\pi}\left((\frac{2\pi}{3} - 0) + 2\arg\left(\frac{1 - ze^{-i2\pi/3}}{1 - ze^{0}}\right)\right. \\
&\quad + e^{i2\pi/3}(\frac{4\pi}{3} - \frac{2\pi}{3}) + 2e^{i2\pi/3}\arg\left(\frac{1 - ze^{-i4\pi/3}}{1 - ze^{-i2\pi/3}}\right) \\
&\quad \left. + e^{i4\pi/3}(2\pi - \frac{4\pi}{3}) + 2e^{i4\pi/3}\arg\left(\frac{1 - ze^{-2\pi i}}{1 - ze^{-i4\pi/3}}\right)\right) \\
&= \frac{2\pi}{3(2\pi)}\left(1 + e^{i2\pi/3} + e^{i4\pi/3}\right) \\
&\quad + \frac{1}{\pi}\left(\arg\left(\frac{1 - ze^{-i2\pi/3}}{1 - ze^{0}}\right) + e^{i2\pi/3}\arg\left(\frac{1 - ze^{-i4\pi/3}}{1 - ze^{-i2\pi/3}}\right) + e^{i4\pi/3}\arg\left(\frac{1 - ze^{-2\pi i}}{1 - ze^{-i4\pi/3}}\right)\right) \\
&= 0 + \frac{1}{\pi}\left(\arg\left(\frac{1 - ze^{-i2\pi/3}}{1 - z}\right) + e^{i2\pi/3}\arg\left(\frac{1 - ze^{-i4\pi/3}}{1 - ze^{-i2\pi/3}}\right) + e^{i4\pi/3}\arg\left(\frac{1 - z}{1 - ze^{-i4\pi/3}}\right)\right).
\end{aligned}$$

Figure 5.8 shows the image of the unit disk as graphed in *ComplexTool*. Notice that it appears to be one-to-one on the interior of the unit disk. It certainly is not one-to-one on the boundary! (Entering this formula into *ComplexTool* is a bit unwieldy, so this function is one of the `Pre-defined functions`, the one called `Harmonic Triangle`. We will soon use the *PolyTool* applet, as described on the next page, to graph other similar functions.)

EXERCISE 5.32. Find a general formula that maps the unit disk harmonically to the interior of a convex regular n-gon. ***Try it out!***

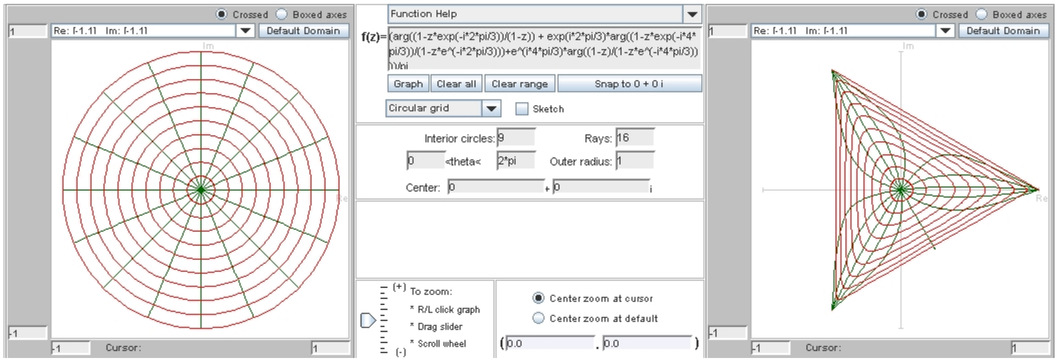

FIGURE 5.8. *ComplexTool* image of the harmonic function mapping to the triangle

SMALL PROJECT 5.33. Refer to Chapter 4 and its discussion of the shear construction. Find the pre-shears of the polygonal mappings in exercise 5.32. In other words, what analytic function do you shear to get that polygonal function? A good first step is to determine the dilatation of this function. See [**4**] for more details.

EXERCISE 5.34. For a non-convex example, consider the function that maps quarters of the unit circle to the four vertices $\{1, i, -1, \frac{i}{2}\}$. Verify that this function is

$$\frac{3i}{8} + \frac{1}{\pi}\left(\arg\left(\frac{1+iz}{1-z}\right) + i\arg\left(\frac{1+z}{1+iz}\right) - \arg\left(\frac{1-iz}{1+z}\right) + \frac{i}{2}\arg\left(\frac{1-z}{1-iz}\right)\right).$$

When we graph this function in *ComplexTool*, we notice that it appears to NOT be one-to-one. Furthermore, $f(0) = \frac{3i}{8}$, so that the image of $\mathbb{D}$ is not the interior of the polygon. This function is another one of the `Pre-defined functions` in *ComplexTool*. ***Try it out!***

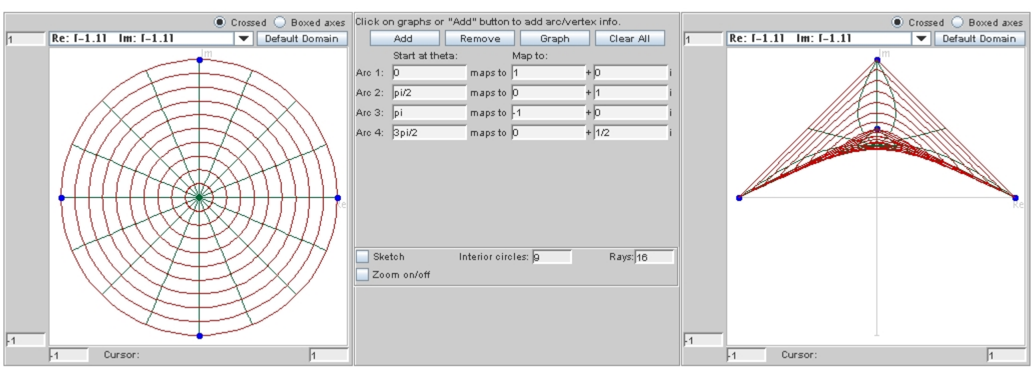

FIGURE 5.9. The *PolyTool* Applet

At this point, you should start using the *PolyTool* applet. In this applet, you can specify which arcs of the unit circle will map to which points in the range, and the applet will compute and graph the harmonic function defined by extending that boundary correspondence to a function on $\mathbb{D}$. When you first open this applet, you see a circle on the left and a blank screen on the right. You can create a harmonic function that maps portions of the boundary of the circle to vertices of a polygon in one of two ways. First, you can click on the unit circle in the left panel to denote an arc endpoint, and continue choosing arc endpoints there, and then choose the target vertices by clicking in the right graph. (Note that as you click, text boxes in between the panels fill with information about where you clicked.) Once you have the boundary correspondence you want, click the `Graph` button. Alternatively, you can click the button that says `Add` to get text boxes for input. For example, to create the function in Exercise 5.34, click `Add`, then fill in the first row of boxes for `Arc 1:` with `0 maps to 1+0i`. When you want another set of arcs, click `Add` again. Note that the `Arc` boxes denote the starting point of the arc (i.e. for the arc from 0 to $\pi/3$, use 0 in the `Arc` box). Continue filling, and when you are ready to compute the Poisson integral to get the harmonic function, click the `Graph` button. Once you have a function graphed, you can "drag" around either the arc endpoints (in the domain on the left) or the target points (in the range on the right) and watch the function dynamically change.

Exploration 5.35. Are there ways of rearranging the boundary conditions to make the function created in Exercise 5.34 univalent? For example, what if the bottom half of the unit circle gets mapped to $i/2$, and the top half of the unit circle is divided into thirds for the other three vertices? This isn't univalent, but in some sense is closer to univalent than the mapping defined in Exercise 5.34. Is there a modification to be made so that it is univalent? ***Try it out!***

Exercise 5.36. This is an extension of Exploration 5.35. Sheil-Small [**17**] proved (by techniques other than those discussed so far) that the harmonic extension of the boundary correspondence below maps the unit disk univalently onto the desired shape:

arc from	to	maps to
$-i$	i	i
i	$-3/5+4i/5$	-1
$-3/5+4i/5$	$-3/5-4i/5$	$i/2$
$-3/5-4i/5$	$-i$	1

For this function, first convince yourself that it appears to be univalent, and then find the function $f(z)$. ***Try it out!***

We will be working a lot with harmonic functions that are extensions of a piecewise constant boundary correspondence, as in the example above. To have a framework for future discussions, we make the following formal definition.

Definition 5.37. Let $\{e^{it_k}\}$ be a partition of $\partial\mathbb{D}$, where $t_0 < t_1 < \ldots < t_n = t_0 + 2\pi$. Let $\hat{f}(e^{it}) = v_k$ for $t_{k-1} < t < t_k$. We call the harmonic extension of this step function (as defined by the Poisson integral formula) $f(z)$.

EXAMPLE 5.38. To better understand the definition, we demonstrate how the notation in Definition 5.37 is used for the function in Exercise 5.36. Since the arc from $-i$ to i can be thought of as the arc along the unit circle from $e^{-i\pi/2}$ to $e^{-i\pi/2}$, we say that $t_0 = -\pi/2$ and $t_1 = \pi/2$. These points map to the vertex at i, so $v_1 = i$. The next arc set is a little more difficult, because we need to find the angle t_2 that goes with the point in the plane $z = \frac{-3}{5} + \frac{4}{5}i = e^{it_2}$. Unfortunately, we can only get a numerical estimate of the angle, found by $\pi + \arctan\left(\frac{4/5}{-3/5}\right) = \pi + \arctan(-4/3) \approx 2.2143$. (We add π because the output of arctangent is always in the first or fourth quadrant, while the angle in question is in the second quadrant.) Thus we have $t_2 \approx 2.2143$ and $v_2 = -1$. Continuing this process, we have $t_3 \approx 4.0689$ and $v_3 = i/2$. Then we finish with $t_4 = 3\pi/2$ and $v_4 = 1$. Notice that as in the definition, $t_4 = t_0 + 2\pi$. Then the function $f(z)$ from Definition 5.37 is the harmonic function that appears to be univalent when graphed in *PolyTool*.

EXERCISE 5.39. Combine the result of Exercise 73 (on page 339) with Definition 5.37 to show that the function $f(z)$ in Definition 5.37 can be written

$$\begin{aligned} f(z) \quad = \quad & v_1\left(\frac{t_1 - t_0}{2\pi}\right) + \frac{v_1}{\pi}\arg\left(\frac{1 - ze^{-it_1}}{1 - ze^{-it_0}}\right) \\ & + v_2\left(\frac{t_2 - t_1}{2\pi}\right) + \frac{v_1}{\pi}\arg\left(\frac{1 - ze^{-it_2}}{1 - ze^{-it_1}}\right) \\ & + \ldots + v_n\left(\frac{t_n - t_{n-1}}{2\pi}\right) + \frac{v_n}{\pi}\arg\left(\frac{1 - ze^{-it_n}}{1 - ze^{-it_{n-1}}}\right). \end{aligned}$$

Try it out!

Since the Poisson integral formula gives rise to a harmonic function, we must learn some of the basics of the theory of harmonic functions before proceeding too far.

5.4. Harmonic Function Theory

Chapter 4 gives a detailed explanation of harmonic functions, as does [**5**]. Much of that material will be helpful for our future investigations, so we repeat it here.

5.4.1. The Basics. Any harmonic function f can be written as $f = h + \overline{g}$, where h and g are analytic functions. The analytic dilatation $\omega(z) = \frac{g'(z)}{h'(z)}$ is, in some sense, a measure of how much the harmonic function does not preserve angles. A dilatation of $\omega(z) \equiv 0$ means that the function is analytic, so must be conformal. A dilatation with modulus near 1 indicates that the function distorts angles greatly. (For more intuition about the dilatation, read Section 4.6 of Chapter 4.) A result of Lewy states that a harmonic function has nonzero Jacobian (denoted $J_f(z) = |h'|^2 - |g'|^2$) if it is locally univalent. This result is in line with our understanding of the relationship between local univalence and a nonvanishing derivative for analytic functions.

THEOREM 5.40 (Lewy's Theorem). For a harmonic function f defined on a domain Ω, if f locally univalent in Ω, then $J_f(z) \neq 0$ for all $z \in \Omega$.

Note that this is equivalent to Lewy's Theorem in Chapter 4.

A nice consequence of Lewy's Theorem is that if a function is locally univalent in Ω, then its analytic dilatation either satisfies $|\omega(z)| < 1$ for all $z \in \Omega$, or $|\omega(z)| > 1$ for all $z \in \Omega$. In our work, we will only study functions that are locally univalent in some domain Ω and satisfy $|\omega(z)| < 1$ for all $z \in \Omega$. These functions are called *sense-preserving* because they preserve the orientation of curves in Ω.

EXERCISE 5.41. Verify that the condition that $J_f(z) \neq 0$ is equivalent to $|\omega(z)| \neq 1$, as long as $h'(z) \neq 0$. Conclude that a function that is locally univalent and sense-preserving must have $J_f(z) > 0$ and $|\omega(z)| < 1$. ***Try it out!***

Of particular interest in this setting is determining how to split up the argument function (which is harmonic and sense-preserving) into h and $\overline{g}$.

EXERCISE 5.42. Show that the function $f(z) = K \arg(z)$ has canonical decomposition $h(z) = \dfrac{1}{2i} K \log(z)$ and $g(z) = \dfrac{1}{2i} \overline{K} \log(z)$. ***Try it out!***

Another consequence of the canonical decomposition of a harmonic function is that we can write the analytic functions h and g defined in some domain Ω in terms of their power series expansions, centered at some $z_0 \in \Omega$, as

$$f(z) = a_0 + \sum_{k=n}^{\infty} a_k (z - z_0)^k + \overline{b_0 + \sum_{k=m}^{\infty} b_k (z - z_0)^k}. \tag{74}$$

If f is sense-preserving, then we necessarily have that either $m > n$ or $m = n$ with $|b_n| < |a_n|$. In either case, when f is represented by Equation 74, we say that f has a zero of order n at z_0.

EXERCISE 5.43. In this exercise, we prove that the zeros of a sense-preserving harmonic function are isolated.

(a) Assume that $f(z)$ is a sense-preserving locally univalent function with series expansion as given in Equation 74. Show that if $f(z_0) = 0$, there exists a positive δ and a function ψ such that, for $0 < |z - z_0| < \delta$ we can write

$$f(z) = h(z) + \overline{g(z)} = a_n (z - z_0)^n (1 + \psi(z)) \tag{75}$$

where

$$\psi(z) = \frac{a_{n+1}}{a_n}(z - z_0) + \frac{a_{n+2}}{a_n}(z - z_0)^2 + \cdots + \overline{b_m (z - z_0)^m} \frac{1}{a_n (z - z_0)^n} + \cdots .$$

(b) Show that part (a) implies that $|\psi(z)| < 1$ for z sufficiently close to z_0, since $m \geq n$ and $|b_n / a_n| < 1$ if $m = n$.

(c) Show that part (b) implies that the zeros of a sense-preserving harmonic function are isolated, since $f(z) \neq 0$ near z_0 (except, of course, at z_0).

Try it out!

5.4.2. The Argument Principle.

5.4.2.1. *Analytic Argument Principle.* The Argument Principle for analytic functions gives a very nice way to describe the number of zeros and poles inside a contour. We take time to explore this topic, even though it is in many introductory complex analysis courses, to emphasize the geometric nature of the result. We first provide a formal definition of a common phrase, the winding number of the image of a contour about the origin.

DEFINITION 5.44. The winding number of the image under $f(z)$ of a simple closed contour Γ about the origin is the net change in argument of $f(z)$ as z traverses Γ in the positive (counterclockwise) direction, divided by 2π. It can be denoted by $\frac{1}{2\pi}\Delta_\Gamma \arg f(z)$.

To explore the relationship between the winding number of the image of a contour about the origin and the number of zeros and poles contained within that contour, do the following exploration.

EXPLORATION 5.45. Open *ComplexTool*. Change the `Interior circles` to `1` and the `Rays` to `0`. This will graph just the boundary of the circle of interest. As you graph the following functions, examine the image of the circle "winds around," or encloses, the origin. Count how many times the image of the circle winds around the origin, making sure that you count the counterclockwise direction as positive and the clockwise direction as negative. If the image of the circle winds around the origin once, you know that there must be a zero of $f(z)$ inside that circle. (Think about this last sentence and make sure you understand it.)

- Graph $f(z) = z^2$, using a circle of radius 1. Use the "Sketch" button and trace around the circle in the domain to get a good feeling for how many times its image winds around the origin. You should already know the answer. (Any other radius will work too. Why is that?)
- Graph $f(z) = z(z-0.3)$. Use circles of radius $0.2, 0.3$ and 0.5. (You may have to zoom in on the image of the circle of radius 0.2 to really understand what it is doing.) If you want, you may check the `Vary radius` checkbox and use the slider to change the radius of the circle.
- Graph $f(z) = z^4 - 6z + 3$. Use circles of radius $0.9, 1.5, 1.7$, and 2.
- Graph $f(z) = \frac{z^4-6z+3}{z-1}$, using circles of radius 0.9 and 1.5.
- Graph $f(z) = \frac{z^4-6z+3}{(z-1)^2}$, using circles of radius $0.9, 1.5$ and 2.
- Go back through the previous 3 items, now changing the function while keeping the radius fixed.
- For all of the previous functions, find the locations of all of the zeros and poles, paying particular attention to how far they are from the origin.
- Make up your own function and do some more experiments.

Based on the explorations above, what is your connection between the winding number of the image of a circle about the origin and the number of zeros and poles inside the circle?

Try it out!

THEOREM 5.46 (Argument Principle for Analytic Functions). Let C be a simple closed contour lying entirely within a domain D. Suppose f is analytic in D except at a finite number of poles inside C and that $f(z) \neq 0$ on C. Then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = N_0 - N_p,$$

where N_0 is the total number of zeros of f inside C and N_p is the total number of poles of f inside C. In determining N_0 and N_p, zeros and poles are counted according to their order or multiplicity.

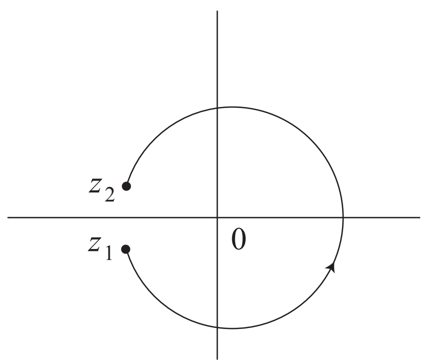

FIGURE 5.10. A path around a branch cut

Before proving Theorem 5.46, we explore the connection between the winding number and the integral $\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz$. To see this connection, we start with a related integral, $\int_{z_1}^{z_2} \frac{f'(z)}{f(z)}dz$, where z_1 and z_2 are points very close to each other, but lying on opposite sides of a branch cut of $\log f(z)$, and we take a "counterclockwise" path along C from z_1 to z_2. (See Figure 5.10.) Now we do the following computation:

$$\begin{aligned}\int_{z_1}^{z_2} \frac{f'(z)}{f(z)}dz &= \log f(z)|_{z_1}^{z_2} \\ &= \ln|f(z_2)| - \ln|f(z_1)| + i(\arg(f(z_2)) - \arg(f(z_1))).\end{aligned}$$

When we take the limit as $z_1 \to z_2$, we get that $\ln|f(z_2)| - \ln|f(z_1)| \to 0$ and $i(\arg(f(z_2)) - \arg(f(z_1))) \to 2\pi i \cdot$ (winding number). (Think carefully about this last statement and make sure you understand it.)

PROOF. The proof of the Argument Principle relies on the Cauchy integral formula and deformation of contours. Take a moment to review these important concepts. We begin by deforming the contour C to a series of smaller contours around the isolated zeros and poles of f. If there are no zeros or poles, then $\frac{f'(z)}{f(z)}$ is analytic, so the integral is zero, as desired. We then analyze the zeros and poles individually, and add the results together to get the desired conclusion. More formally, when f has zeros or poles inside C, they must be isolated, and because f is analytic on C, there are only a finite number of distinct zeros or poles inside C. Denote the zeros and poles by z_j, for $j = 1, 2, \ldots, n$. Let γ_j be a circle of radius $\delta > 0$ centered at z_j, where δ is chosen small enough that the circles γ_j all lie in D and do not meet each other. Join each circle γ_j to C by a Jordan arc λ_j in D. Consider the closed path Γ formed by moving around C in the positive (counterclockwise) direction while making a detour along each λ_j to γ_j, running once around this circle in the clockwise (negative) direction, then returning along λ_j to C. See Figure 5.11.

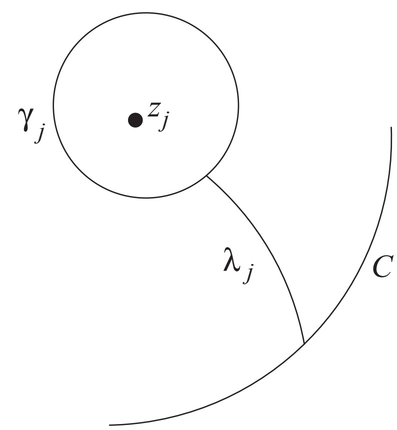

FIGURE 5.11. The contour Γ

This curve Γ contains no zeros or poles of f, so $\Delta_\Gamma \arg f(z) = \dfrac{1}{2\pi i} \oint_\Gamma \dfrac{f'(z)}{f(z)} dz = 0$ by the argument above. When considering the total change in argument along Γ of $f(z)$, the contributions of the arcs λ_j along Γ cancel out, so that

$$\Delta_C \arg f(z) = \sum_{j=1}^{n} \Delta_{\gamma_j} \arg f(z),$$

where each of the circles γ_j is now traversed in the positive (counterclockwise) direction. Thus now we may consider each individual γ_j and sum the results.

Suppose that f has a zero of order n at $z = z_j$. Then $f(z) = (z - z_j)^n f_n(z)$, where $f_n(z)$ is an analytic function satisfying $f_n(z_j) \neq 0$. (If you can't remember why this is true, look in any standard introductory complex analysis book.) Then

$$f'(z) = n(z - z_j)^{n-1} f_n(z) + (z - z_j)^n f_n'(z)$$

and

$$\begin{aligned} \frac{f'(z)}{f(z)} &= \frac{n(z - z_j)^{n-1} f_n(z) + (z - z_j)^n f_n'(z)}{(z - z_j)^n f_n(z)} \\ &= \frac{n}{z - z_j} + \frac{f_n'(z)}{f_n(z)}. \end{aligned}$$

Now we note that when we integrate the above expression along γ_j, we get $n(2\pi i) + 0$, because $\frac{f_n'(z)}{f_n(z)}$ is analytic inside the contour.

Now suppose that f has a pole of order m at $z = z_k$. This means that f can be rewritten as $f(z) = (z - z_k)^{-m} f_m(z)$, where f_m is analytic an nonzero at $z = z_k$. Then, as previously, we have

$$\begin{aligned} \frac{f'(z)}{f(z)} &= \frac{-m(z - z_k)^{-m-1} f_m(z) + (z - z_k)^{-m} f_m'(z)}{(z - z_k)^{-m} f_m(z)} \\ &= \frac{-m}{z - z_k} + \frac{f_m'(z)}{f_m(z)}. \end{aligned}$$

Once again, when we integrate the above expression along γ_k, we get $-m(2\pi i) + 0$.

Summing over $j = 1, 2, \ldots n$ gives us the integral over Γ and the desired result.

□

5.4.2.2. *Argument Principle for Harmonic Functions.* There are many versions of the argument principle for harmonic functions. We only need the simple proof presented in this section, developed by Duren, Hengartner, and Laugesen ([**6**]).

Theorem 5.47 (Argument Principle for Harmonic Functions). Let D be a Jordan domain with boundary C. Suppose f be a sense-preserving harmonic function on D, continuous in $\overline{D}$ and $f(z) \neq 0$ on C. Then $\Delta_C \arg f(z) = 2\pi N$, where N is the total number of zeros of $f(z)$ in D, counted according to multiplicity.

Proof. First, we suppose that f has no zeros in D. This means that $N = 0$ and the origin is not an element of $f(D \cup C)$. A fact from topology says that in this case, $\Delta_C \arg f(z) = 0$, and the theorem is proved. We will prove this fact. Let ϕ be a homeomorphism of the closed unit square S onto $D \cup C$ with the restriction of ϕ to the boundary, $\hat{\phi} : \partial S \to C$, also a homeomorphism. See Figure 5.12.

The composition $F = f \circ \phi$ is a continuous mapping of S onto the plane with no zeros, and we want to prove that $\Delta_{\partial S} \arg F(z) = 0$. Begin by subdividing S into finitely many small squares S_j so that on each S_j, the argument of F varies by at most $\pi/2$. Then $\Delta_{\partial S_j} \arg F(z) = 0$ (since $F(S_j)$ cannot enclose the origin). Now when we consider $\Delta_{\partial S} \arg F(z)$, it is the sum $\sum_j \Delta_{\partial S_j} \arg F(z)$ because the contributions to

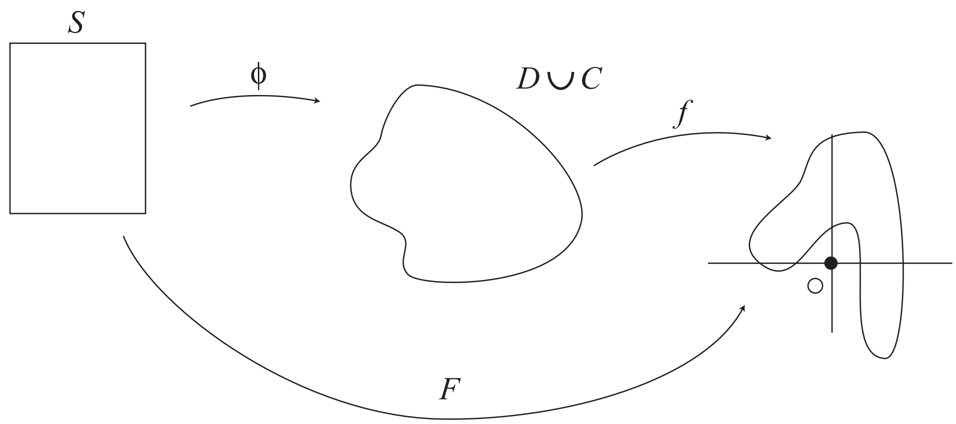

FIGURE 5.12. The composition of f and ϕ.

the sum from the boundaries of each S_j cancel out, except where the boundary of S_j agrees with the boundary of S. Thus $\Delta_{\partial S} \arg F(z) = 0$, as desired.

Now consider the case where f does have zeros in D. Because the zeros are isolated (as proven in Exercise 5.43), and because f is not zero on C, there are only a finite number of distinct zeros in D. We proceed in a manner similar to the proof of the analytic argument principle, and, denote the zeros by z_j, for $j = 1, 2, \ldots, n$. Let γ_j be a circle of radius $\delta > 0$ centered at z_j, where δ is chosen so small that the circles γ_j all lie in D and do not meet each other. Join each circle γ_j to C by a Jordan arc λ_j in D. Consider the closed path Γ formed by moving around C in the positive direction while making a detour along each λ_j to γ_j, running once around this circle in the clockwise (negative) direction, then returning along λ_j to C. (See Figure 5.11 on page 348.) This curve Γ contains no zeros of f, so $\Delta_\Gamma \arg F(z) = 0$ by the first case in this proof. When considering the total change in argument along Γ of $f(z)$, the contributions of the arcs λ_j along Γ cancel out, so that

$$\Delta_C \arg f(z) = \sum_{j=1}^{n} \Delta_{\gamma_j} \arg f(z),$$

where each of the circles γ_j is now traversed in the positive (counterclockwise) direction. Thus now we may consider each individual γ_j and sum the results.

Now suppose that f has a zero of order n at a point z_0. Then, as observed in Exercise 5.43 on page 345, f can be locally written as

$$f(z) = a_n(z - z_0)^n(1 + \psi(z))$$

where $a_n \neq 0$ and $|\psi(z)| < 1$ on a sufficiently small circle γ defined by $|z - z_0| = \delta$. This shows that

$$\Delta_\gamma \arg f(z) = n\Delta_\gamma \arg(z - z_0) + \Delta_\gamma \arg(1 + \psi(z)) = 2\pi n. \tag{76}$$

Therefore, if f has zeros of order n_j at the points z_j, the conclusion is that

$$\Delta_C \arg f(z) = \sum_{j=1}^{n} \Delta_{\gamma_j} \arg f(z) = 2\pi \sum_{j=1}^{n} n_j = 2\pi N,$$

and the theorem is proved.

□

EXERCISE 5.48. Justify why $\Delta_\gamma \arg(1+\psi(z)) = 0$ in Equation 76. ***Try it out!***

For the next section, we shift our focus back to the polygonal maps defined in Section 5.3. We will be using the Argument Principle for Harmonic Functions in the proof of the Rado-Kneser-Choquet Theorem.

5.5. Rado-Kneser-Choquet Theorem

As you examine the image of the unit disk using the examples in Section 5.3, you may notice that some of the functions seem to be one-to-one on the interior of the domain, while others do not seem to be univalent. Look again at the examples, and compare functions which map to convex domains versus functions that map to non-convex domains.

EXPLORATION 5.49. Make a conjecture about when functions are one-to-one, using the exercises from Section 5.3 as a springboard. Do this before reading the Rado-Kneser-Choquet Theorem! ***Try it out!***

In general, we completely understand the behavior of harmonic extensions (as defined in Definition 5.37) that map to convex regions:

THEOREM 5.50 (Rado-Kneser-Choquet Theorem). Let Ω be a subset of $\mathbb{C}$ that is a bounded convex domain whose boundary is a Jordan curve Γ. Let $\hat{f}$ map $\partial\mathbb{D}$ continuously onto Γ and suppose that $\hat{f}(e^{it})$ runs once around Γ monotonically as e^{it} runs around $\partial\mathbb{D}$. Then the harmonic extension given in the Poisson integral formula is univalent in $\mathbb{D}$ and defines a harmonic mapping of $\mathbb{D}$ onto Ω.

For the proof of this important theorem, we use the following lemma.

LEMMA 5.51. Let ψ be a real-valued function harmonic in $\mathbb{D}$ and continuous in $\mathbb{D}$. Suppose ψ has the property that, after a rotation of coordinates, $\psi(e^{it}) - \psi(e^{-it}) \geq 0$ on the interval $[0, \pi]$, with strict inequality $\psi(e^{it}) - \psi(e^{-it}) > 0$ on some subinterval $[a, b]$ with $0 \leq a < b \leq \pi$. Then ψ has no critical points in $\mathbb{D}$.

The condition on ψ seems a bit mysterious at first, and so we should discuss it. One kind of function for which this property holds is a ψ that is at most bivalent on $\partial\mathbb{D}$. What does "at most bivalent" mean? We know that univalent means that a function is one-to-one. Bivalent means that a function is two-to-one, or that there may be $z_1 \neq z_2$ such that $f(z_1) = f(z_2)$, but that if $f(z_1) = f(z_2) = f(z_3)$, then at least 2 of z_1, z_2, z_3

must be equal. Alternatively, another kind of function ψ described in Lemma 5.51 is one that is continuous on $\partial\mathbb{D}$ where $\psi(e^{it})$ rises from a minimum at $e^{-i\alpha}$ to a maximum at $e^{i\alpha}$, then decreases again to its minimum at $e^{-i\alpha}$ as e^{it} runs around the unit circle, without having any other local extrema, but allowing arcs of constancy. See Figure 5.13.

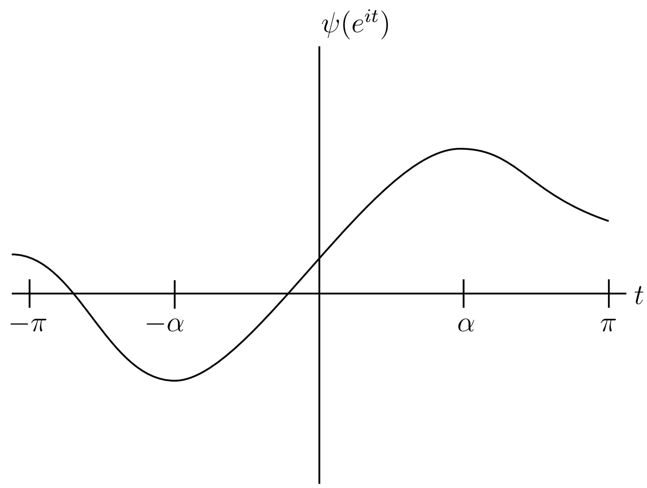

FIGURE 5.13. Boundary condition of one possible ψ satisfying the hypotheses of Lemma 5.51.

PROOF OF LEMMA 5.51. To show that ψ has no critical points in $\mathbb{D}$, we must show that $\frac{\partial\psi}{\partial z} \neq 0$ in $\mathbb{D}$. This is equivalent to saying that

$$\frac{1}{2}\left(\frac{\partial\psi}{\partial x} - i\frac{\partial\psi}{\partial y}\right) \neq 0.$$

At this point, we will simplify the proof by simply proving that $\psi_z(0) \neq 0$, and claim that will be sufficient. Indeed, if z_0 is some other point in $\mathbb{D}$, consider the function $\varphi(z) = \frac{z_0 - z}{1 - \overline{z_0} z}$ that is a conformal self-map of $\mathbb{D}$ with $\varphi(0) = z_0$, and consider the composition $F(\zeta) = \psi(\varphi(\zeta))$. Observe that F is harmonic in $\mathbb{D}$, continuous in $\overline{\mathbb{D}}$, and satisfies the same condition about $F(e^{it}) - F(e^{-it})$ as ψ does. Applying the chain rule to $F(\zeta)$ gives that $F_\zeta(\zeta) = \psi_z(\varphi(\zeta))\varphi'(\zeta)$, since φ is analytic and thus has $\varphi_{\bar{\zeta}} = 0$. (In general, the chain rule is more complicated for harmonic functions. Here, since φ is analytic, the chain rule takes its familiar form.) Plugging in 0 for ζ gives $F_\zeta(0) = \psi_z(z_0)\varphi'(0)$, implying that if $F_\zeta(0) = 0$ then also $\psi_\zeta(z_0) = 0$. Thus when we have proven that $\psi_z(0) \neq 0$, we will be able to generalize to $\psi_z(z_0) \neq 0$ for all z_0 in $\mathbb{D}$.

Now we use the Poisson integral formula to prove that $\psi_z(0) \neq 0$. Substituting in ψ (or $\hat{\psi}(e^{it}) = \lim_{r\to 1} \psi(re^{it})$ on $\partial\mathbb{D}$) gives

$$\psi(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2}\hat{\psi}(e^{it})dt = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-z\,\overline{z}}{(e^{it}-z)(e^{-it}-\ \overline{z})}\hat{\psi}(e^{it})dt.$$

When we differentiate both sides with respect to z, the integral depends only on t, so we are just left differentiating the integrand. Doing this, we find

$$\begin{aligned}
\frac{\partial}{\partial z}\left(\hat{\psi}(e^{it})\frac{1-z\,\overline{z}}{(e^{it}-z)(e^{-it}-\ \overline{z})}\right) &= \frac{\hat{\psi}(e^{it})}{e^{-it}-\ \overline{z}}\frac{\partial}{\partial z}\left(\frac{1-z\,\overline{z}}{e^{it}-z}\right) \\
&= \left(\frac{\hat{\psi}(e^{it})}{e^{-it}-\ \overline{z}}\right)\cdot\left(\frac{e^{it}(e^{-it}-\ \overline{z})}{(e^{it}-z)^2}\right) \\
&= \hat{\psi}(e^{it})\left(\frac{e^{it}}{(e^{it}-z)^2}\right),
\end{aligned}$$

leading to the conclusion that

$$\psi_z(0) = \frac{1}{2\pi}\int_0^{2\pi} \hat{\psi}(e^{it})e^{-it}dt.$$

From the hypotheses of the lemma, we know that there is some $t \in (0,\pi)$ such that $\psi(e^{it}) - \psi(e^{-it}) > 0$. Thus

$$\begin{aligned}
\operatorname{Im}\psi_z(0) &= \operatorname{Im}\left(\frac{1}{2\pi}\int_0^{2\pi} \hat{\psi}(e^{it})e^{-it}dt\right) \\
&= -\frac{1}{2\pi}\int_0^{2\pi} \hat{\psi}(e^{it})\sin(t)dt \\
&= -\frac{1}{2\pi}\left(\int_0^{\pi} \hat{\psi}(e^{it})\sin(t)dt + \int_{-\pi}^{0} \hat{\psi}(e^{it})\sin(t)dt\right) \text{ since } \hat{\psi} \text{ is periodic} \\
&= -\frac{1}{2\pi}\left(\int_0^{\pi} \hat{\psi}(e^{it})\sin(t)dt - \int_0^{\pi} \hat{\psi}(e^{-it})\sin(t)dt\right) \\
&= -\frac{1}{2\pi}\int_0^{\pi}(\hat{\psi}(e^{it}) - \hat{\psi}(e^{-it}))\sin(t)dt < 0.
\end{aligned}$$

The last inequality relies on the fact that $\sin(t)$ is non-negative on the interval $[0,\pi]$. We have now shown that $\operatorname{Im}\psi_z(0) \neq 0$, thus proving the lemma.

□

Proof of Theorem 5.50. Without loss of generality, assume that $\hat{f}(e^{it})$ runs around Γ in the counterclockwise direction as t increases. (Otherwise, take conjugates.) We will show that if the function f is not locally univalent in $\mathbb{D}$, then Lemma 5.51 will give a contradiction.

Suppose that $f = u + iv$ is not locally univalent, or that the Jacobian of f vanishes at some point z_0 in $\mathbb{D}$. This means that the matrix $\begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix}$ has a determinant of 0 at z_0. From linear algebra, we know that this means that the system of equations

$$\begin{aligned} au_x + bv_x &= 0 \\ au_y + bv_y &= 0 \end{aligned}$$

has a nonzero solution (a, b). Thus the real-valued harmonic function $\psi = au + bv$ has a critical point at z_0 (since $(a, b) \neq (0, 0)$). However, the hypothesis of Theorem 5.50 implies that ψ satisfies the hypothesis of Lemma 5.51. Thus we have a contradiction, so f must be locally univalent.

Now that we see that f is locally univalent, we apply the argument principle to show that f is univalent in $\mathbb{D}$. Since f is sense-preserving on $\partial\mathbb{D}$ and locally univalent, f is sense-preserving throughout $\mathbb{D}$. Now, if f is not univalent, there are two points z_1 and z_2 in $\mathbb{D}$ such that $f(z_1) = f(z_2)$. However, that would imply that the function $f(z) - f(z_1)$ has two zeros in $\mathbb{D}$, so that the winding number of $f(z) - f(z_1)$ about the origin is 2, which contradicts the hypotheses about the boundary correspondence. This completes the proof.

□

Exercise 5.52. Give a detailed proof of the statement, "However, the hypothesis of Theorem 5.50 implies that ψ satisfies the hypothesis of Lemma 5.51." ***Try it out!***

Notice that the description of $\hat{f}$ in Theorem 5.50 does not require that it be one-to-one on $\partial\mathbb{D}$, but permits arcs of constancy. Furthermore, the Rado-Kneser-Choquet Theorem is actually true in the case where $\hat{f}$ has jump discontinuities, as long as the image of $\partial\mathbb{D}$ is not contained in a straight line. This requires some additional justification, so we state it separately as a corollary.

Corollary 5.53. Let $f(z)$ be defined as in Definition 5.37 on 343. Suppose the vertices $v_1, v_2, \ldots v_n$, when traversed in order, define a convex polygon, with the interior of the polygon denoted by Ω. Then the function $f(z)$ is univalent in $\mathbb{D}$ and defines a harmonic mapping from $\mathbb{D}$ onto Ω.

Here is some intuition behind the proof of Corollary 5.53. Consider a sequence of functions $\hat{f}_m(e^{it})$ that are continuous and converge to the boundary correspondence $\hat{f}(e^{it})$ of Definition 5.37. (One possible such sequence of functions can be described by having $\hat{f}_m(e^{it}) = v_k$ for t-values in the interval $(t_{k-1} + \frac{t_k - t_{k-1}}{2m}, t_k - \frac{t_k - t_{k-1}}{2m})$ while for t-values in the interval $(t_k - \frac{t_k - t_{k-1}}{2m}, t_k + \frac{t_{k+1} - t_k}{2m})$, $\hat{f}_m(e^{it})$ maps the interval linearly to the segment between v_k and v_{k+1}.) Each of these functions $\hat{f}_m(e^{it})$ satisfies the conditions of the Rado-Kneser-Choquet Theorem, so extends to a univalent harmonic function, $f_m(z)$, in the unit disk. But the functions f_m converge uniformly on compact subsets of $\mathbb{D}$, so the entire sequence converges uniformly to f in $\mathbb{D}$. Therefore, $f(z)$ inherits the univalence from the sequence. The fact that the limit function is still univalent is not immediately apparent–full details may be found in [**8**].

Interestingly enough, this theorem does not guarantee anything about univalence if the domain Ω is not convex. In fact, the expectation is that univalence will not be achieved. For example, look at Exercise 5.34 on page 342.

Exploration 5.54. Extend the explorations begun in Exploration 5.35 on page 343. Now, instead of modifying the boundary correspondence, start with the correspondence in Exercise 5.36. Then, move the vertex that is at $i/2$, moving it closer to i. A very nice picture comes from having the vertex set be $\{1, i, -1, \frac{9i}{10}\}$. In this last we see the lack of univalence very clearly. ***Try it out!***

5.5.1. Boundary behavior.

In this section, we explore what seems to be true with some of the above examples: There appears to be some very interesting boundary behavior of our harmonic extensions of step functions. Examine this behavior in the following exploration.

Exploration 5.55. Using *ComplexTool* or *PolyTool*, graph the function from $\mathbb{D}$ to a triangle in Example 5.31. Now investigate the behavior of the boundary using the sketching tool of the applet. In particular, approach the break point between arcs (such as $z = 1$) along different paths. First approach radially, then approach along a line that is not a radius of the circle. Observe how these different paths that approach 1 cause the image of the path to approach different points along the line segment that makes up a portion of the boundary of the range. (As you get very close to an arc endpoint, the image of the sketch may jump to a vertex–here, examine where the image is immediately before that jump.) Technology hint: in *PolyTool* and *ComplexTool*, you can hit the `Graph` button to clear all previous sketching but keep the polygonal map. Repeat this exercise with some of the other examples of polygonal functions. Try to answer some of the following questions:

(1) Given a point ζ on the boundary of the polygon, is it possible to find a path γ approaching $\partial\mathbb{D}$ such that $\gamma(z)$ approaches ζ?
(2) As you approach an arc endpoint in $\partial\mathbb{D}$ radially, what point on the boundary of the polygon do you approach?

Try it out!

As you performed the exploration above, you probably discovered some of the known properties of the boundary behavior of harmonic extensions of step functions. These results were originally proven by Hengartner and Schober [**8**], who proved a more general form of the theorem below. We now restate their theorem as it applies to the step functions of Definition 5.37. In the theorem below, the *cluster set* of f at a point e^{it_k} is the set of all possible limits of sequences $\{z_n\}$, where z_n are inside Γ, and $\lim_{n\to\infty}(z_n) = e^{it_k}$.

THEOREM 5.56. Let f be the harmonic extension of a step function $\hat{f}(e^{it_k})$ in Definition 5.37. Denote by Γ the polygon defined by the vertices v_k. By definition, the radial limits $\lim_{r\to 1} f(re^{it})$ lie on Γ for all t except those in the set $\{t_k\}$. Then the unrestricted limit

$$\hat{f}(e^{it}) = \lim_{z\to e^{it}} f(z)$$

exists at every point $e^{it} \in \partial\mathbb{D}\backslash\{e^{it_1}, e^{it_2}, ..., e^{it_n}\}$ and lies on Γ. Furthermore,

(1) the one-sided limits as $t \to t_k$ are

$$\lim_{t\to t_k^-} \hat{f}(e^{it}) = v_k \qquad \text{and} \qquad \lim_{t\to t_k^+} \hat{f}(e^{it}) = v_{k+1}; \text{ and}$$

(2) the cluster set of f at each point $e^{it_k} \in \{e^{it_1}, e^{it_2}, ..., e^{it_n}\}$ is the linear segment joining v_k to v_{k+1}.

PROOF. Part (1) of the theorem follows directly from the definition of the function f and from properties of the Poisson integral formula. That is, since the boundary correspondence is defined in Definition 5.37, the limits follow. Now we need to show part (2). Now let us consider e^{it_k}. If z approaches e^{it_k} along the circular arc

$$\arg\left(\frac{e^{it_k}+z}{e^{it_k}-z}\right) = \frac{\lambda\pi}{2}, \qquad -1 < \lambda < 1, \tag{77}$$

then $f(z)$ converges to the value

$$\frac{1}{2}(1-\lambda)v_k + \frac{1}{2}(1+\lambda)v_{k+1}.$$

Therefore the cluster set of f at t_k is the line segment joining v_k and v_{k+1}, and part (2) is proven.

□

EXERCISE 5.57. Use basic ideas from analytic geometry to observe that equation 77 is a circular arc. (Hint: First consider the cases where z is either on the circle or is the center of the circle for some intuition.) ***Try it out!***

EXERCISE 5.58. It is important to note that Theorem 5.56 holds for even non-univalent mappings. Go back to some of the previous examples and identify the line segment that connnects the vertices. In particular, regraph the example from Exercise 5.34. Using the `Sketch` utility in either *ComplexTool* or *Polytool*, check that the limit as you approach one of the t_k does appear to be that line segment. ***Try it out!***

5.6. Star Mappings

From the Rado-Kneser-Choquet Theorem, we see that harmonic functions mapping the unit disk $\mathbb{D}$ to convex polygons are well-understood. That is, if we define a harmonic function mapping from the unit disk to a convex polygon as in Definition 5.37, the function is univalent in $\mathbb{D}$. Theorem 5.56 describes the boundary behavior fully, showing that the limit of the function as we approach one of the break points between vertex pre-images, t_k, gives the line segment joining the vertices.

However, non-convex polygons are not nearly as well-understood. We first examine non-convex polygons in their simplest mathematical form: the ones of regular stars.

Definition 5.59. By an n-pointed "star," or "r-star", we mean an equilateral $2n$-gon with the vertex set,

$$\left\{r\alpha^{2k}, \alpha^{2k+1} : k = 1, 2, \ldots, n \text{ and } \alpha = e^{i\pi/n}\right\},$$

where r is some real constant.

Notice that when $r = 1$, the n-pointed star is a regular $2n$-gon, and when $r < \cos(\pi/n)$ or $r > \sec(\pi/n)$, the star is a strictly non-convex $2n$-gon. Our preimages of the vertices of the $2n$-gon will be arcs *centered* at the $2n$th roots of unity (this is different from our previous examples).

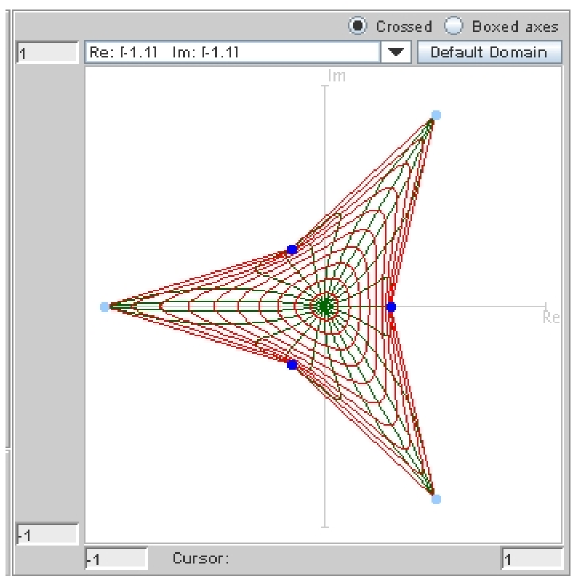

Figure 5.14. The 0.3-star for $n = 3$

Example 5.60. We will find a harmonic mapping of the unit disk into the 0.3-star. More precisely, will find the harmonic extension of the following boundary correspondence.

for t from	to	$\hat{f}(e^{it})$
$-\pi/6$	$\pi/6$	0.3
$\pi/6$	$\pi/2$	$e^{i\pi/3}$
$\pi/2$	$5\pi/6$	$0.3e^{i2\pi/3}$
$5\pi/6$	$7\pi/6$	-1
$7\pi/6$	$3\pi/2$	$0.3e^{i4\pi/3}$
$3\pi/2$	$11\pi/6$	$e^{i5\pi/3}$

After going through details similarly to previous examples, we discover the harmonic extension is

$$\begin{aligned} f(z) &= \frac{1}{\pi}\left[0.3\arg\left(\frac{1-ze^{-i\pi/6}}{1-ze^{i\pi/6}}\right)\right. \\ &\quad +e^{i\pi/3}\arg\left(\frac{1+iz}{1-ze^{-i\pi/6}}\right)+0.3e^{i2\pi/3}\arg\left(\frac{1-ze^{-i5\pi/6}}{1+iz}\right) \\ &\quad -\arg\left(\frac{1-ze^{-i7\pi/6}}{1-ze^{-i5\pi/6}}\right)+0.3e^{i4\pi/3}\arg\left(\frac{1-iz}{1-ze^{-7i\pi/6}}\right) \\ &\quad \left.+e^{i5\pi/3}\arg\left(\frac{1-ze^{-i11\pi/6}}{1-ze^{-i3\pi/2}}\right)\right]. \end{aligned}$$

Graph this function using *ComplexTool* (it is one of the `Pre-defined functions`). Notice that it appears to be univalent. We certainly have not yet proved its univalence.

Exercise 5.61. Prove that if $f(z)$ is the harmonic extension to the r-star as defined in Definition 5.59, then $f(0) = 0$. Interpret this result geometrically. ***Try it out!***

Exercise 5.62. Modify the function in Exercise 5.60 to have $r = 0.15$ and see whether it appears univalent. To graph this new function in *ComplexTool*, choose the previous star as one of the `Pre-defined functions` and then modify the equation that shows in the function box. ***Try it out!***

To work with these stars, we may sometimes want to vary the boundary correspondence. That is, we may want to not split up $\partial\mathbb{D}$ completely evenly among the $2n$ vertices. It will become useful to us to have an unequal correspondence in the boundary arcs, but maintain some symmetry. To do this, we will still consider arcs centered at the $2n$-th roots of unity, but alternating between larger and smaller arcs. If we examine the geometry of this matter, we realize that an even split would make each arc have length $\frac{2\pi}{2n} = \pi/n$. Two consecutive arcs would together have length $2\pi/n$. To still maintain some symmetry, but let the arcs alternate in size, we want two consecutive arcs to still add to $2\pi/n$, but not split evenly. We introduce the parameter p, with $0 < p < 1$, as a tool to explain how the arcs are split. We will want two consecutive arcs split into $p2\pi/n$ and $(1-p)2\pi/n$. Note that the sum is still $2\pi/n$. This is formally described in the definition below.

DEFINITION 5.63. Let $n \geq 2$ be a fixed integer, r be a positive real number, and $\alpha = e^{i\pi/n}$. Define a boundary correspondence for all but a finite number of points on $\partial\mathbb{D}$ by mapping arcs with endpoints $\{\alpha e^{-ip\pi/n}, \alpha e^{ip\pi/n}, 0 \leq k \leq n-1\}$ as follows:

$$\hat{f}(e^{it}) = \begin{cases} r\alpha^{2k}, & e^{it} \in (\alpha^{2k}e^{-ip\pi/n}, \alpha^{2k}e^{ip\pi/n}) \\ \alpha^{2k+1}, & e^{it} \in (\alpha^{2k+1}e^{-i(1-p)\pi/n}, \alpha^{2k+1}e^{i(1-p)\pi/n}) \end{cases} . \tag{78}$$

Let f be the Poisson extension of $\hat{f}$.

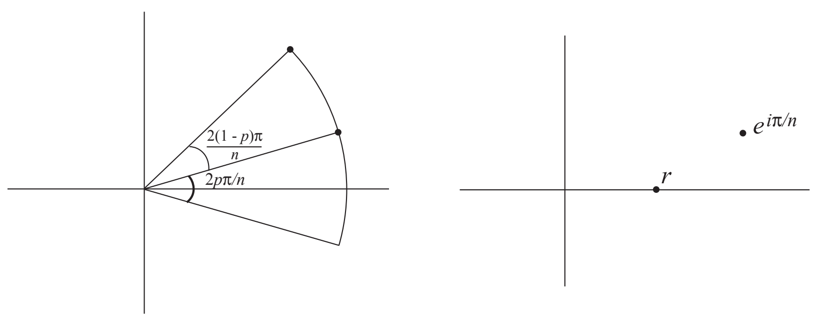

FIGURE 5.15. The first two arcs and their images according to Definition 5.63. The dots on the left-hand-side indicate points of discontinuity of the boundary correspondence.

Note that the arc $(\alpha^{2k}e^{-ip\pi/n}, \alpha^{2k}e^{ip\pi/n})$ centered at α^{2k} is mapped to the vertex $r\alpha^{2k}$ and the arc $(\alpha^{2k+1}e^{-i(1-p)\pi/n}, \alpha^{2k+1}e^{i(1-p)\pi/n})$ centered at α^{2k+1} is mapped to the vertex α^{2k+1}.

EXERCISE 5.64. Show that the interval in the second half of Equation 78,

$$(\alpha^{2k+1}e^{-i(1-p)\pi/n}, \alpha^{2k+1}e^{i(1-p)\pi/n}),$$

can be written more compactly as $(\alpha^{2k}e^{ip\pi/n}, \alpha^{2k+2}e^{-ip\pi/n})$. ***Try it out!***

At this point, you should start using with the *StarTool* applet. The default for this applet is the 3-pointed star discussed in Example 5.60. Note that the arcs and their target vertices are color-coded (with a light blue arc mapping to a light blue vertex, for example). The default p-value is 0.5, which corresponds to evenly spaced arcs. You can use the slider bars (the plus/minus buttons for n) or type in the text boxes to change the values for n, p, and r. The maximum n-value allowed by the applet is $n = 18$, which is sufficient for the explorations below. As with *ComplexTool*, there is the option to `Sketch` on the graph to get a better feel for the mapping properties of these stars. There is also an option to `Show roots of` $\omega(z)$. The roots of $\omega(z)$ will be helpful in future discussion, but are not essential for the starting explorations. In general, try to first get a good feel for what happens for "small" values of n, such as $4, 5, 6,$ or 7.

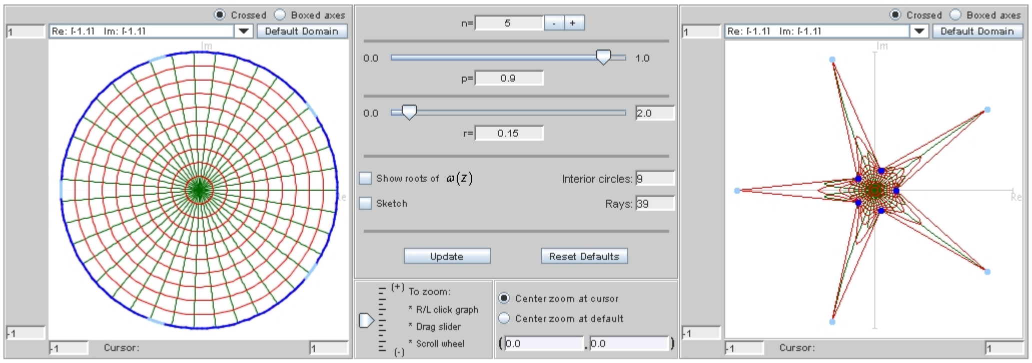

FIGURE 5.16. A star with $n = 5$, $r = 0.15$, and $p = 0.9$

EXPLORATION 5.65. Do some explorations with different values of n and r. See if you can find a pattern for univalence of the star function. Here are some avenues for exploration.

(1) What is the relationship between the r value you choose and the p-value necessary for univalence? Is there a range of p that works?
(2) What happens as p goes to 0 or 1?
(3) For a given p-value, can you determine the "minimal" r? That is, how small can you make r and maintain univalence?
(4) Is there a minimum r-value, one for which there is no p-value that will achieve univalence?
(5) For a fixed r, as you change n, what happens to the p that you need to achieve univalence?
(6) What is the full relationship between r, n, and p? (It is unlikely that you will answer this question now, but make some conjectures about it.)

Try it out!

5.7. Dilatations of Polygonal Maps are Blaschke Products

We are now ready to use the tools of harmonic functions to study the polygonal maps. To understand when the star maps are univalent, we must first examine their dilatation. As we discover in this section, the dilatation of a polygonal map is always in the form of a Blaschke product.

EXERCISE 5.66. Consider the function generated in Exercise 5.34 on page 342. Complete all of the following:

(a) Find the formulas for $h(z)$ and $g(z)$. Use the result of Exercise 5.42 for this computation.

(b) Find the derivatives $h'(z)$ and $g'(z)$, verifying that the derivatives simplify to

$$h'(z) = \frac{1}{(2\pi i)(1-z^4)}\left((\frac{3}{2}-\frac{3}{2}i)z^2 - 3iz + (\frac{5}{2}+\frac{5}{2}i)\right)$$

and $g'(z) = \dfrac{1}{(2\pi i)(1-z^4)}\left((\dfrac{5}{2}-\dfrac{5}{2}i)z^2 + 3iz + (\dfrac{3}{2}+\dfrac{3}{2}i)\right)$.

(c) Show that the zeros of $g'(z)$ are $z_1 \approx 0.9245 - 0.9245i$ and $z_2 \approx -0.3245 + 0.3245i$, and that the zeros of $h'(z)$ are $1/\overline{z_1}$ and $1/\overline{z_2}$. Thus we are able to write the dilatation as

$$\omega(z) = C\left(\frac{z_1 - z}{1 - \overline{z_1}z}\right)\left(\frac{z_2 - z}{1 - \overline{z_2}z}\right),$$

where C is some constant.

(d) What are $|z_1|, |z_2|$, and $|C|$? These will be helpful to recall as we look ahead to Theorem 5.81 in Section 5.8.

Try it out!

Motivated by the results of Exercise 5.66, we now examine functions that are of a particular form.

Exploration 5.67. We examine the properties of functions of the form $B_{z_0}(z) = \dfrac{z_0 - z}{1 - \overline{z_0}z}$.

- Using *ComplexTool*, graph the image of the unit disk under the functions $B_{0.5}(z) = \frac{0.5-z}{1-0.5z}$, $B_{0.5\exp(i\pi/4)}(z) = \frac{0.5e^{i\pi/4}-z}{1-0.5e^{-i\pi/4}z}$, and $B_{0.5i}(z) = \frac{0.5i-z}{1+0.5iz}$. What is $B(0)$? What is the image of the unit disk under $B(z)$? Does $B(z)$ appear to be univalent?
- Now graph the image of the unit disk under the function $B_{2\exp(i\pi/4)}(z) = \frac{2e^{i\pi/4}-z}{1-2e^{-i\pi/4}z}$. What is $B(0)$? What is the image of the unit disk under $B(z)$? Does $B(z)$ appear to be univalent?
- If we consider $B_{z_0}(z)$, can we determine $B_{z_0}(0)$ in general? What effect does $\arg(z_0)$ have on the location of $B_{z_0}(0)$? What effect does $|z_0|$ have on the image of the unit disk?
- Now let's multiply the functions $B_{z_0}(z)$. First graph the image of the unit disk under $f_1(z) = (B_{0.5}(z))^2 = \left(\frac{0.5-z}{1-0.5z}\right)^2$. What is $f(0)$? What is the image of the unit disk under $f(z)$? Does $f(z)$ appear to be univalent? How does this compare to the function $f(z) = z^2$?
- Now graph the image of the unit disk under $f_2(z) = B_{0.5}(z)B_{0.5i}(z)$, $f_3(z) = B_{0.5}(z)B_{0.2i}(z)$, and $f_4(z) = B_{0.5\exp(i\pi/4)}(z)B_{0.2i}(z)$. What are $f_2(0)$, $f_3(0)$, and $f_4(0)$? How do $f_2(0)$, $f_3(0)$, and $f_4(0)$ relate to the values of z_0 in the functions $B_{z_0}(z)$?

Try it out!

Definition 5.68. A Blaschke factor is $B_{z_0}(z) = \frac{z_0 - z}{1 - \overline{z_0} z}$, and a finite Blaschke product or order n is a product of n Blaschke factors, possibly multiplied by a constant ζ such that $|\zeta| = 1$:

$$\zeta \prod_{k=1}^{n} \frac{z_k - z}{1 - \overline{z_k} z}.$$

Note that the multiplication by ζ is simply a rotation.

Remark 5.69. It is important to note that the Blashke product definition given above is a bit non-standard. The standard definition of Blaschke product, as given in Chapters 4 and 6, assumes that $|z_k| < 1$. Here we do not place that restriction on z_k for the purpose of simplifying our computations.

In this section, we use the result of Exercise 5.39 on page 344 to see that the dilatation of the harmonic polygonal functions to an n-gon is a Blaschke product of order at most $n - 2$. This result was proven by T. Sheil-Small in [**17**], and is also discussed in [**5**].

We use the notation $f_k(z)$ to denote the contribution to the function $f(z)$ that arises from applying the Poisson integral formula to the boundary correspondence for $t_{k-1} < t < t_k$. We then have $f(z) = \sum_{k=1}^{n} f_k(z)$. On the interval $t_{k-1} < t < t_k$, we observe that

$$f_k(z) = \frac{v_k}{2\pi}(t_k - t_{k-1}) + \frac{v_k}{\pi} \arg\left(\frac{1 - ze^{-it_k}}{1 - ze^{-it_{k-1}}}\right)$$

by Definition 5.37. We now consider the canonical decomposition of $f_k(z) = h_k(z) + \overline{g_k(z)}$. By Exercise 5.42, we have

$$\begin{aligned} h_k(z) &= \frac{v_k}{2\pi}(t_k - t_{k-1}) + \frac{v_k}{2\pi i} \log\left(\frac{1 - ze^{-it_k}}{1 - ze^{-it_{k-1}}}\right) \\ &= \frac{v_k}{2\pi}(t_k - t_{k-1}) + \frac{v_k}{2\pi i} \left(\log(1 - ze^{-it_k}) - \log(1 - ze^{-it_{k-1}})\right) \end{aligned}$$

and

$$\begin{aligned} g_k(z) &= \frac{\overline{v_k}}{2\pi i} \log\left(\frac{1 - ze^{-it_k}}{1 - ze^{-it_{k-1}}}\right) \\ &= \frac{\overline{v_k}}{2\pi i} \left(\log(1 - ze^{-it_k}) - \log(1 - ze^{-it_{k-1}})\right). \end{aligned}$$

The computations that follow will give some rigor to our intuition: since h and g are sums of logarithms, their derivatives are sums of terms that have $1 - ze^{-it_k}$ in the denominators. We will combine these factors, and hope that we can see each derivative as a Blaschke product. Now, when we look at $h(z) = \sum_{k=1}^{n} h_k(z)$ and take derivatives, we see that:

$$h'(z) = \sum_{k=1}^{n} \frac{v_k}{2\pi i}\left(\frac{-e^{-it_k}}{1-ze^{-it_k}} - \frac{-e^{-it_{k-1}}}{1-ze^{-it_{k-1}}}\right) \tag{79}$$

$$= \sum_{k=1}^{n} \frac{v_k}{2\pi i}\left(\frac{1}{z-e^{it_k}} - \frac{1}{z-e^{it_{k-1}}}\right). \tag{80}$$

The function $g'(z)$ is identical except for having $\overline{v_k}$ instead of v_k:

$$g'(z) = \sum_{k=1}^{n} \frac{\overline{v_k}}{2\pi i}\left(\frac{-e^{-it_k}}{1-ze^{-it_k}} - \frac{-e^{-it_{k-1}}}{1-ze^{-it_{k-1}}}\right) \tag{81}$$

$$= \sum_{k=1}^{n} \frac{\overline{v_k}}{2\pi i}\left(\frac{1}{z-e^{it_k}} - \frac{1}{z-e^{it_{k-1}}}\right). \tag{82}$$

Combining like factors gives us a more compact form, with

$$h'(z) = \frac{1}{2\pi i}\sum_{k=1}^{n} \frac{v_k - v_{k+1}}{z-e^{it_k}} \qquad \text{and} \qquad g'(z) = \frac{1}{2\pi i}\sum_{k=1}^{n} \frac{\overline{v}_k - \overline{v}_{k+1}}{z-e^{it_k}}. \tag{83}$$

It will be useful for the upcoming discussion to note that $\sum_{k=1}^{n}(v_k - v_{k+1}) = 0$, since $v_{n+1} = v_1$.

EXERCISE 5.70. Prove that $\sum_{k=1}^{n}(v_k - v_{k+1}) = 0$, since $v_{n+1} = v_1$. Interpret this result geometrically.
Try it out!

We rely heavily upon the observation that

$$\overline{h'(1/\overline{z})} = z^2 g'(z) \qquad \text{or } \overline{g'(1/\overline{z})} = z^2 h'(z). \tag{84}$$

Equation 84 arises from the following computation:

$$\begin{aligned}
\overline{h'(1/\overline{z})} - z^2 g'(z) &= \frac{-1}{2\pi i}\sum_{k=1}^{n}\frac{\overline{v}_k - \overline{v}_{k+1}}{1/z - e^{-it_k}} - \frac{z^2}{2\pi i}\sum_{k=1}^{n}\frac{\overline{v}_k - \overline{v}_{k+1}}{z - e^{it_k}} \\
&= \frac{-z}{2\pi i}\sum_{k=1}^{n}\frac{e^{it_k}(\overline{v}_k - \overline{v}_{k+1})}{e^{it_k} - z} - \frac{z^2}{2\pi i}\sum_{k=1}^{n}\frac{\overline{v}_k - \overline{v}_{k+1}}{z - e^{it_k}} \\
&= \frac{z}{2\pi i}\sum_{k=1}^{n}\frac{e^{it_k}(\overline{v}_k - \overline{v}_{k+1})}{z - e^{it_k}} - \frac{z^2}{2\pi i}\sum_{k=1}^{n}\frac{\overline{v}_k - \overline{v}_{k+1}}{z - e^{it_k}} \\
&= \frac{1}{2\pi i}\sum_{k=1}^{n}\frac{(\overline{v}_k - \overline{v}_{k+1})(ze^{it_k} - z^2)}{z - e^{it_k}} \\
&= \frac{1}{2\pi i}\sum_{k=1}^{n}\frac{(\overline{v}_k - \overline{v}_{k+1})(z)(e^{it_k} - z)}{z - e^{it_k}} \\
&= \frac{-1}{2\pi i}\sum_{k=1}^{n}(\overline{v}_k - \overline{v}_{k+1})(z) \\
&= 0.
\end{aligned}$$

Exercise 5.71. We have just proven the first half of Equation 84. Using that result, prove the second part of Equation 84 with minimal calculation. ***Try it out!***

Exercise 5.72. Interpret this result geometrically. That is, note that if we have a value $z_0 \in \mathbb{D}$ such that $g'(z_0) = 0$, then what do we know about the zeros of h'? How are the locations of the zeros of h' related to the locations of the zeros of g'? Completion of this exercise will give some intuition about the proof that is to come. ***Try it out!***

Exercise 5.73. Show that $h'(z)$ and $g'(z)$ of Exercise 5.66 satisfy Equation 84. Relate this to the previous two exercises–do the conclusions of those exercises also hold true for this example? ***Try it out!***

For simplicity of notation, let us consider the functions $h'(z)$ and $g'(z)$. We can already tell that if we got a common denominator for $h'(z)$ or $g'(z)$, that the denominator would be $\prod_{k=1}^{n}(z - e^{it_k})$, and we would guess that the ratio of the two would give us a product of rational functions. At this point, that is all we can tell–it is not obvious that this product should be a Blaschke product, although we may expect it to be from the explorations we did in Exercise 5.66. The remainder of this section will be devoted to determining that this is, indeed, a Blaschke product, as well as finding the order of that product.

Exploration 5.74. Based on the results of Exercise 5.66, and upon other examples in this chapter, make a conjecture about the number of Blaschke factors that should be in the dilatation of a harmonic function from $\mathbb{D}$ to an n-gon. ***Try it out!***

Obtaining a common denominator for both h' and g', we can look at them as

$$h'(z) = \frac{P(z)}{S(z)} \qquad \text{and} \qquad g'(z) = \frac{Q(z)}{S(z)},$$

where $S(z) = \prod_{k=1}^{n}(z - e^{it_k})$. Now we need to consider what P and Q look like. Consider that by brute force, each term of the $P(z)$ looks like $(v_k - v_{k+1}) \prod_{j=1;j\neq k}^{n}(z - e^{it_j})$, or, put more simply, a polynomial of degree at most $n-1$. Let us consider the z^{n-1} term of $P(z)$. It is simply $v_k - v_{k+1}$ for each piece of the sum, so it must be $\sum_{k=1}^{n}(v_k - v_{k+1})$, which we already observed to be 0. Thus we have shown that $P(z)$ has degree at most $n-2$. The same argument works for $Q(z)$, since it has the same structure as $P(z)$ but with conjugates over the v_k.

We now turn our attention to the denominator, which is $S(z) = \prod_{k=1}^{n}(z - e^{it_k})$.

Exercise 5.75. Show that the following equation holds:

$$S(1/\,\overline{z}) = \left(\frac{1}{\overline{z}}\right)^n (-1)^n\, \overline{S(z)} \prod_{k=1}^{n} e^{it_k}.$$

Try it out!

Put another way, we could write

$$\overline{S(1/\,\overline{z})} = \left(\frac{1}{z}\right)^n (-1)^n S(z) \prod_{k=1}^{n} e^{-it_k}. \tag{85}$$

Exercise 5.76. Show that Equation 85 holds for the denominator of the derivatives in Exercise 5.66, $2\pi i(1 - z^4)$. ***Try it out!***

Now we can combine Equations 84 and 85 to get a relationship between $P(z)$ and $Q(z)$. Directly substituting into $\overline{h'(1/\,\overline{z})} = z^2 g'(z)$, we see that

$$\begin{aligned}
\overline{h'(1/\,\overline{z})} &= z^2 g'(z)\\
\frac{\overline{P(1/\,\overline{z})}}{\overline{S(1/\,\overline{z})}} &= z^2\frac{Q(z)}{S(z)}\\
\frac{\overline{P(1/\,\overline{z})}}{\left(\frac{1}{z}\right)^n(-1)^n S(z)\prod_{k=1}^n e^{-it_k}} &= z^2\frac{Q(z)}{S(z)}.
\end{aligned}$$

This leads to the relationship

$$z^{n-2}\ \overline{P(1/\,\overline{z})} = (-1)^n Q(z)\prod_{k=1}^n e^{-it_k}. \tag{86}$$

EXERCISE 5.77. Using the result above, show that

$$z^{n-2}\ \overline{Q(1/\,\overline{z})} = (-1)^n P(z)\prod_{k=1}^n e^{-it_k}. \tag{87}$$

Try it out!

Since the function f is orientation-preserving, we know that $h'(z) \neq 0$ in $\mathbb{D}$. This implies that $P(z) \neq 0$ in $\mathbb{D}$. In particular, $P(0) \neq 0$. Substituting 0 into equation 87, we find that the left hand side must not be zero, which forces the degree of Q to be at least $n-2$. However, we had previously determined that the degree of Q must be at most $n-2$. Thus the degree of Q is $n-2$. Similarly, the degree of P is also $n-2$. Since the degree of Q is $n-2$, let us write

$$Q(z) = z^m\prod_{k=1}^{n-m-2}(z - z_k)$$

to show that Q may have m zeros at the origin and $n-m-2$ zeros elsewhere (note that the z_k need not be distinct). Now using Equation 87, we can write

$$z^{n-2}\ \overline{\left(\frac{1}{\overline{z}}\right)^m\prod_{k=1}^{n-m-2}\left(\frac{1}{\overline{z}} - z_k\right)} = (-1)^n P(z)\prod_{k=1}^n e^{-it_k}.$$

The left hand side of the above equation may be rewritten as

$$z^{n-m-2}\frac{1}{z^{n-m-2}}\prod_{k=1}^{n-m-2}\left(1 - z\ \overline{z_k}\right).$$

At this point we can see that since the zeros of Q are z_k, the zeros of P are the zeros of $\prod_{k=1}^{n-m-2}(1 - z\ \overline{z_k})$, which are precisely $1/\,\overline{z}_k$. Now we are able to see what the Blaschke product is.

EXERCISE 5.78. Find the relationship between the number of zeros of Q and the number of zeros of P. In particular, if Q has degree $n-2$ with m zeros at the origin and $n-m-2$ zeros away from the origin, then how many of the zeros of P are at the origin? How many of the zeros of P are away from the origin? ***Try it out!***

We now summarize the results of our work in this section (as originally proved by T. Sheil-Small, [**17**] Theorem 1; see also [**5**]).

THEOREM 5.79. Let f be the harmonic extension of the step function $\hat{f}(e^{it})$ as given in Definition 5.37. Then

$$g'(z) = \frac{Q(z)}{S(z)} \qquad \text{and} \qquad h'(z) = \frac{P(z)}{S(z)},$$

where $Q(z), P(z)$, and $S(z)$ are defined as above, and P and Q are polynomials of degree at most $n-2$. Furthermore, their ratio $\omega(z)$ satisfies $|\omega(z)| = 1$ when $|z| = 1$, so takes the form of a Blaschke product of degree at most $n-2$.

5.8. An Important Univalence Theorem

In this section, we examine a theorem of Sheil-Small that tells when the harmonic function in Definition 5.37 is univalent. In particular, the location of the zeros of the analytic dilatation $\omega(z) = \frac{g'(z)}{h'(z)}$ are sufficient to tell when the harmonic function is univalent.

EXPLORATION 5.80. Open up the *StarTool* applet. Check the box in front of `Show roots of` $\omega(z)$. You will see extra dots appear in the right-hand pane (the range of the function), as well as a unit circle for reference. These dots denote the locations of the zeros of the dilatation $\omega(z)$. Now experiment with the values of p and r to see if there is a relationship between the roots of $\omega(z)$ and whether the resulting star is univalent. Do this for various values of n to see if your result seems to hold. Does your conjecture agree with the examination of a function that maps $\mathbb{D}$ to a different non-convex polygon, as in Exercise 5.66? ***Try it out!***

The theorem below was proven by Sheil-Small, and is Theorem 11.6.6 of [**18**].

THEOREM 5.81. Let f be a harmonic function of the form in Definition 5.37. Here the function f is the harmonic extension of a piecewise constant boundary function with values on the m vertices of a polygonal region Ω, so that, by Theorem 5.79, the dilatation of f is a Blaschke product with at most $m-2$ factors. Then f is univalent in $\mathbb{D}$ if and only if all zeros of ω lie in $\mathbb{D}$. In this case, f is a harmonic mapping of $\mathbb{D}$ **onto** Ω.

PROOF. First, suppose that f is univalent in $\mathbb{D}$. If a Blaschke factor is defined as $\varphi_{z_0}(z) = \frac{z_0 - z}{1 - \overline{z_0} z}$, with the constant z_0 not having modulus 1, then we notice that since the dilatation ω is a product of a finite number of Blaschke factors, $\omega(z) \neq 0$ on the unit circle. This is because the zero of the Blaschke factor is z_0, and if $|z_0| = 1$, we get that $\varphi_{z_0}(z) = z_0$ for all z. If ω has a zero at some point z_0 outside of $\overline{\mathbb{D}}$, then it has a pole at $1/\overline{z_0} \in \mathbb{D}$. If it also has zeros in $\mathbb{D}$, then there are points in $\mathbb{D}$ where $|\omega(z)| < 1$ and other points where $|\omega(z)| > 1$. This implies that the Jacobian of f changes sign in $\mathbb{D}$, which would force the Jacobian to equal 0 at some point in $\mathbb{D}$, contradicting Lewy's theorem, which says that the Jacobian is non-zero since f is locally univalent. Thus there are only two possibilities for a univalent f: Either all of the zeros of $\omega(z)$ lie in $\mathbb{D}$, or all lie outside $\overline{\mathbb{D}}$. But if the zeros of ω lie outside of $\overline{\mathbb{D}}$, then $|\omega(z)| > 1$ in $\mathbb{D}$ and f has negative Jacobian, contradicting its construction as a sense-preserving boundary function. Therefore, all of the zeros of ω must lie in $\mathbb{D}$.

Conversely, assume all of the zeros of ω lie within $\mathbb{D}$. By the mapping properties of Blaschke products, $|\omega(z)| < 1$ in $\mathbb{D}$. We use the argument principle to show that f is univalent in $\mathbb{D}$ and maps $\mathbb{D}$ onto Ω. Choose an arbitrary point $w_0 \in \Omega$. Let C_ϵ be the path in $\overline{\mathbb{D}}$ consisting of arcs of the unit circle along with small circular arcs of radius ϵ about the points b_k (the points b_k are the arc endpoints in the domain disk), as shown in Figure 5.17.

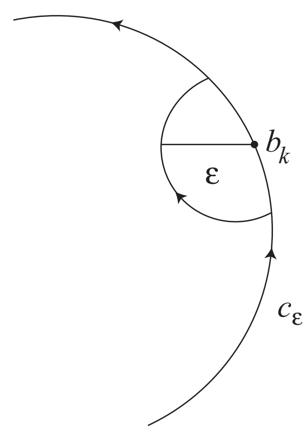

FIGURE 5.17. Tiny circles around the b_k

If ϵ is sufficiently small, the image of C_ϵ will not go through w_0, and will have winding number $+1$ around w_0. Since $|\omega(z)| < 1$ inside C_ϵ, it follows from the argument principle for harmonic functions that $f(z) - w_0$ has one simple zero inside C_ϵ (or, put another way, $f(z) = w_0$ has exactly one solution for $z \in \mathbb{D}$). Thus $\Omega \subset f(\mathbb{D})$. Now do a similar construction with $w_0 \notin \Omega$ to show that $w_0 \notin f(\mathbb{D})$. Thus f maps $\mathbb{D}$ univalently onto Ω. □

To apply this theorem to the star mappings of Section 5.6, we study the dilatation of the star mappings in detail.

5.9. The Dilatation for Star Mappings

In this section, we will use Definition 5.63 of Section 5.6 as the starting point. We will build upon the basic formula for the functions $h'(z)$ and $g'(z)$, and then will simplify the dilatation $\omega(z)$ as a Blaschke product. By doing so, we completely determine which star functions are univalent.

Using Equations (79) and (81) of Section 5.7, along with Definition 5.63, we find the following equations for $h'(z)$ and $g'(z)$ for the star functions:

$$\begin{aligned}
h'(z) &= \frac{r\alpha^0}{2\pi i}\left(\frac{1}{z-\alpha^0 e^{ip\pi/n}} - \frac{1}{z-\alpha^0 e^{-ip\pi/n}}\right) \\
&\qquad + \frac{\alpha}{2\pi i}\left(\frac{1}{z-\alpha^2 e^{-ip\pi/n}} - \frac{1}{z-\alpha^0 e^{ip\pi/n}}\right) \\
&\qquad\quad + \frac{r\alpha^2}{2\pi i}\left(\frac{1}{z-\alpha^2 e^{ip\pi/n}} - \frac{1}{z-\alpha^2 e^{-ip\pi/n}}\right) + \dots \\
&= \frac{r}{2\pi i}\sum_{k=0}^{n-1}\alpha^{2k}\left(\frac{1}{z-\alpha^{2k} e^{ip\pi/n}} - \frac{1}{z-\alpha^{2k} e^{-ip\pi/n}}\right) \\
&\qquad + \frac{1}{2\pi i}\sum_{k=0}^{n-1}\alpha^{2k+1}\left(\frac{1}{z-\alpha^{2k+2} e^{-ip\pi/n}} - \frac{1}{z-\alpha^{2k} e^{ip\pi/n}}\right)
\end{aligned} \tag{88}$$

and

$$\begin{aligned}
g'(z) &= \frac{r}{2\pi i}\sum_{k=0}^{n-1}\overline{\alpha^{2k}}\left(\frac{1}{z-\alpha^{2k} e^{ip\pi/n}} - \frac{1}{z-\alpha^{2k} e^{-ip\pi/n}}\right) \\
&\qquad + \frac{1}{2\pi i}\sum_{k=0}^{n-1}\overline{\alpha^{2k+1}}\left(\frac{1}{z-\alpha^{2k+2} e^{-ip\pi/n}} - \frac{1}{z-\alpha^{2k} e^{ip\pi/n}}\right).
\end{aligned} \tag{89}$$

Our goal is to express $\omega(z) = g'(z)/h'(z)$ as the ratio of Blaschke products guaranteed by Sheil-Small's work. To that end, we first establish a few general algebraic identities involving sums of the quantities of the type found in the expansions of $h'(z)$ and $g'(z)$ above.

It is a basic complex identity that if ζ is a primitive mth root of unity, then

$$\prod_{k=1}^{m}(z-\zeta^k) = z^m - 1. \tag{90}$$

EXERCISE 5.82. Prove Equation (90). Interpret this result geometrically. ***Try it out!***

EXERCISE 5.83. Again, let ζ be a primitive mth root of unity. Show that

$$\prod_{k=0}^{m-1}(z-\zeta^k a) = z^m - a^m. \tag{91}$$

Hint: Replace z with z/a in Equation (90). ***Try it out!***

Now we work to answer the hard question: How do we add together all of the sums in equations (88) and (89), given that their numerators are not simply the constant 1? As an intermediate step toward achieving this, we establish the following identity.

LEMMA 5.84. If ζ is a primitive mth root of unity, then

$$\sum_{k=0}^{m-1}\frac{\zeta^k}{z-\zeta^k a} = \frac{ma^{m-1}}{z^m-a^m}. \tag{92}$$

EXERCISE 5.85. Prove the lemma, using the following steps in the partial fraction decomposition.

(1) Recall Equation (91) and note how it fits in with this formula.
(2) Note that since we have n distinct linear factors in the denominator, we can expect to find that

$$\frac{ma^{m-1}}{z^m-a^m} = \sum_{k=0}^{m-1}\frac{a_k}{z-\zeta^k a}.$$

(3) We will find an arbitrary a_{k_0}. By setting $z=\zeta^{k_0}a$, establish that

$$a_{k_0} = \frac{ma^{m-1}}{\prod_{k\neq k_0}(\zeta^{k_0}a-\zeta^k a)} = \frac{m}{\prod_{k\neq k_0}(\zeta^{k_0}-\zeta^k)}.$$

(4) Show that $\displaystyle\prod_{k\neq k_0}(\zeta^{k_0}-\zeta^k) = m\zeta^{-k_0}$. It will be helpful to remember that ζ^{k_0} is an mth root of unity, so $\zeta^{k_0 m}=1$.
(5) Conclude that $a_{k_0}=\zeta^{k_0}$.
(6) Equation (92) should follow.

Try it out!

We now recall the result of Exercise 5.64 on page 359 that

$$\alpha^{2k+2}e^{-ip\pi/n} = \alpha^{2k+1}e^{i(1-p)\pi/n} \qquad \text{and} \qquad \alpha^{2k}e^{ip\pi/n} = \alpha^{2k+1}e^{-i(1-p)\pi/n}.$$

Combining with the earlier work, we have

$$\frac{1}{2\pi i}\sum_{k=0}^{n-1}\frac{\alpha^{2k+1}}{z-\alpha^{2k+2}e^{-ip\pi/n}} \;=\; \frac{-1}{2\pi i}\left(\frac{ne^{i(1-p)(n-1)\pi/n}}{z^n+e^{i(1-p)\pi}}\right) \qquad \text{and}$$

$$-\frac{1}{2\pi i}\sum_{k=0}^{n-1}\frac{\alpha^{2k+1}}{z-\alpha^{2k}e^{ip\pi/n}} \;=\; \frac{1}{2\pi i}\left(\frac{ne^{-i(1-p)(n-1)\pi/n}}{z^n+e^{-i(1-p)\pi}}\right).$$

EXERCISE 5.86. Prove that

$$-\frac{1}{2\pi i}\sum_{k=0}^{n-1}\frac{\alpha^{2k+1}}{z-\alpha^{2k}e^{ip\pi/n}} = \frac{1}{2\pi i}\left(\frac{ne^{-i(1-p)(n-1)\pi/n}}{z^n+e^{-i(1-p)\pi}}\right).$$

Try it out!

Combining all of these together, we see that

$$h'(z) \;=\; \frac{n}{2\pi i}\left(\frac{re^{i(\frac{n-1}{n})p\pi}}{z^n-e^{ip\pi}}+\frac{-re^{-i(\frac{n-1}{n})p\pi}}{z^n-e^{-ip\pi}}\right.$$

$$\left.+\frac{-e^{i(\frac{n-1}{n})(1-p)\pi}}{z^n+e^{i(1-p)\pi}}+\frac{e^{-i(\frac{n-1}{n})(1-p)\pi}}{z^n+e^{-i(1-p)\pi}}\right). \tag{93}$$

We need to keep our goal in mind: We know from Theorem 5.79 that $\frac{g'(z)}{h'(z)}$ can be written as a Blaschke product. To do this, we will have to find a common denominator and combine the four terms of $h'(z)$ to see the quotient. As an initial step, we find that we can write the common denominator more simply than it first appears.

EXERCISE 5.87. Prove that $(z^n-e^{ip\pi})(z^n-e^{-ip\pi}) = (z^n+e^{i(1-p)\pi})(z^n+e^{-i(1-p)\pi})$. We will call this product $S_n(z)$. ***Try it out!***

EXERCISE 5.88. Using basic algebra (finding a common denominator, simplifying, and using properties of $z+\overline{z}$), prove that

$$h'(z) \;=\; \frac{n}{\pi S_n(z)}\left(z^n\left(r\sin\left(\frac{(n-1)p\pi}{n}\right)-\sin\left(\frac{(n-1)(1-p)\pi}{n}\right)\right)\right.$$

$$\left.+r\sin\left(\frac{p\pi}{n}\right)+\sin\left(\frac{(1-p)\pi}{n}\right)\right). \tag{94}$$

Try it out!

Through methods similar to those of the simplification of $h'(z)$, we can also prove that

$$g'(z) \;=\; \frac{nz^{n-2}}{\pi S_n(z)}\left(z^n\left(r\sin\left(\frac{p\pi}{n}\right)+\sin\left(\frac{(n-1)(1-p)\pi}{n}\right)\right)\right.$$

$$\left.+r\sin\left(\frac{(n-1)p\pi}{n}\right)-\sin\left(\frac{(n-1)(1-p)\pi}{n}\right)\right). \tag{95}$$

With this simplified form of $g'(z)$, we use

$$c = \frac{\sin\frac{(n-1)(1-p)\pi}{n} - r\sin\frac{(n-1)p\pi}{n}}{r\sin\frac{p\pi}{n} + \sin\frac{(1-p)\pi}{n}} \tag{96}$$

to write

$$g'(z) = \frac{nz^{n-2}}{\pi}\left(r\sin\left(\frac{p\pi}{n}\right) + \sin\left(\frac{(1-p)\pi}{n}\right)\right)\frac{z^n - c}{S_n(z)}. \tag{97}$$

Now we are ready to pull together the result of Theorem 5.81 and the dilatation that we just simplified. When the zeros of this dilatation are within the unit disk, then the harmonic function $f = h + \overline{g}$ that defines the star is univalent. By a straightforward computation, we find that the dilatation of f is

$$\omega(z) = \frac{z^{n-2}(z^n - c)}{1 - z^n c}. \tag{98}$$

EXERCISE 5.89. Look again at Theorem 5.81 and verify that it does, indeed, hold for the star function. ***Try it out!***

EXPLORATION 5.90. Notice that f is univalent when $|c| < 1$. Using that observation, do the following:

- Use the *StarTool* applet to explore graphically what relationship there is between n, p, and c.
- For a fixed n, find the range of p-values that make $|c| < 1$.
- For a fixed p, find the range of n-values that make $|c| < 1$.

Try it out!

LARGE PROJECT 5.91. If you move just one vertex of the star, do the same results hold for the relationship between n and p? (For example, take the vertex at r, and move it to $r + \epsilon$ or $r - \epsilon$. Is the star still univalent?)

EXERCISE 5.92. For a given n, consider the formula for c in Equation 96 to be a function of p alone. Prove that any star configuration is possible; that is, prove that for any value of r, a value of p can be found to make $|c| < 1$. What ranges of p makes this happen? Conversely, prove that for all values of $r < \cos(\pi/n)$ or $r > \sec(\pi/n)$, a p can be chosen to make the function NOT univalent. Why is this not true for $\cos(\pi/n) \leq r \leq \sec(\pi/n)$? For more information, see Theorem 4 in [**7**]. ***Try it out!***

SMALL PROJECT 5.93. Refer to Chapter 2. For what values of c is the dilatation a perfect square? Find and describe the associated minimal surfaces. Can these surfaces be described as examples of other well-known surfaces? For more information on this project, see [**12**].

5.10. Open Questions

Large Project 5.94. Can we map to any polygon univalently? The star setup takes full advantage of the symmetry. Once you lose that advantage, it is much harder to discover whether the zeros of the dilatation have modulus less than 1. This question is known as the Mapping Problem, proposed by T. Sheil-Small in [**17**].

Small Project 5.95. Look at a function f that is not univalent. Now look at the set $S \subset \mathbb{D}$ of points on which the function f is univalent. First, how do you find that set? What is the shape of S? Is it starlike? Is it convex? Is it connected? Is it simply connected? Can $\mathbb{D} \backslash S$ be connected?

Small Project 5.96. In this chapter, we discussed one way of proving that a harmonic function is univalent by looking at zeros of the analytic dilatation $\omega(z)$. In Chapter 4, there is another set of criteria for univalence, as demonstrated in Section 4.6. Connect these two avenues of investigation. For example, does one imply the other? How does the work with stars in this chapter generalize to the approach in Chapter 4? Are there results in this chapter that could not be found using the methods of Chapter 4?

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