CHAPTER 2

Soap films, Differential Geometry, and Minimal Surfaces

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2.1. Introduction

Minimal surfaces are beautiful geometric objects with interesting properties that can be studied with the help of computers. Some standard examples of minimal surfaces in $\mathbb{R}^3$ are the plane, Enneper's surface, the catenoid, the helicoid, and Scherk's doubly periodic surface (see Figure 2.1; note that the images shown are just part of these surfaces and that each surface actually continues on forever).

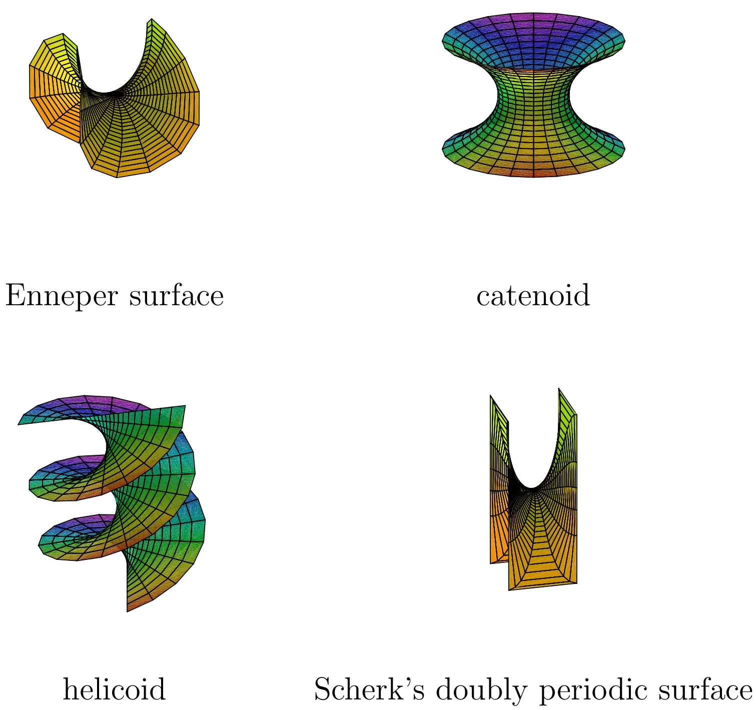


Figure 2.1. Examples of some minimal surfaces

Minimal surfaces are related to soap films that result when a wire frame is dipped in soap solution. To get a sense of this connection, consider the following problem.

Steiner Problem: Four houses are located so that they form the vertices of a square that has sides of length one mile. These neighbors want to connect their houses with a road of least length. What should the shape of the road be?

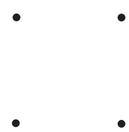
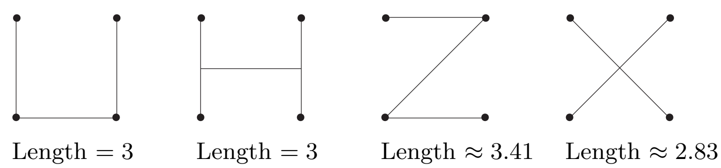


FIGURE 2.2. What is the shortest path connecting these 4 vertices?

Some possible solutions include the following:



However, none of these is the solution. The correct solution has a length of $1+\sqrt{3} \approx 2.7$ miles (see Figure 2.3). For more information about Steiner problems see [**5**] or [**13**].

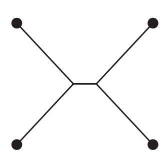
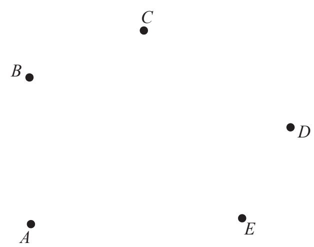
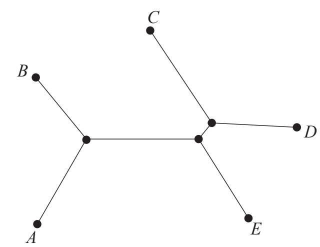


FIGURE 2.3. The shortest path connecting these 4 vertices.

How can we generalize this problem? One way is to have n-vertices. So the problem becomes, given n cities find a connected system of straight line segments of shortest length such that any pair of cities is connected by a path of line segments.



A n-vertices Steiner problem



The solution

Another way to generalize this idea is to move up a dimension. What is the analogue of the Steiner problem in one dimension higher? The Steiner Problem minimizes distance (1-dimensional object) in a plane (2-dimensional object). Soap film minimizes area (2-dimensional object) in space (3-dimensional object).

Is there a connection between these two- and three-dimensional optimization problems? Consider the soap film created by dipping a cube frame into soap solution shown in Figure 2.4. The soap film creates the minimum possible surface area for a surface with a cube as its boundary. If we projected this surface onto the plane, the resulting shape would look like the solution to the Steiner problem with four vertices shown in Figure 2.3.

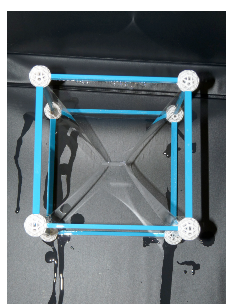


FIGURE 2.4. Soap film formed by a cube.

Water molecules exert a force on each other. Near the surface of the water there is a greater force pulling the molecules toward the center of the water. This force creates surface tension which tends to minimize the surface area of the shape. Soap solution has a lower surface tension than water and this permits the formation of soap films which also tend to minimize geometric properties such as length and area. For more information along this line see [**22**].

Minimal surfaces can be created by dipping wire frames into soap solution.

EXAMPLE 2.1. By dipping into soap solution a wire frame of a "slinky" (or helix) with a straw connecting the ends of the "slinky", we can create part of the minimal surface known as the helicoid.

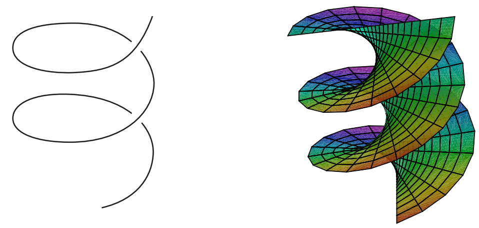


FIGURE 2.5. The wire frame of a slinky can be used to create part of the helicoid.

EXAMPLE 2.2. By dipping a 3-dimensional version of the wire frame shown below (a box frame missing two parallel edges on the top and two parallel edges on the bottom) into soap solution we can create part of the minimal surface known as Scherk's doubly periodic surface.

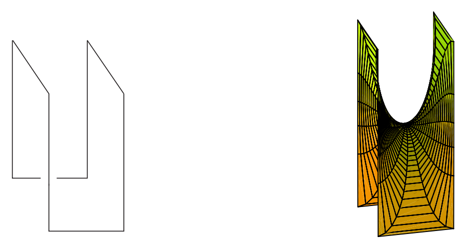


FIGURE 2.6. The wire frame of a box missing 4 edges can be used to create part of Scherk's doubly periodic surface.

EXPLORATION 2.3. Each of the minimal surfaces shown in Figure 2.1 can be formed by dipping a wire frame in soap solution. Determine the shape of the wire frame that creates: (a) Enneper's surface; and (b) the catenoid.

Try it out!

REMARK 2.4. To get a soap film of the part of Enneper's surface shown in Figure 2.1, we can dip a wire frame that matches the seams along a baseball. In fact, dipping such a wire frame in soap solution produces *two* minimal surfaces. The first is one half of the sphere of a baseball and the other is the complementary half of the "baseball." What is interesting is that if you start with one, you can deform it into the other by slightly and carefully blowing air into the soap film. There is a third, mysterious and unseen minimal surface one passes through while doing this, and this minimal surface is unstable. In other words, it cannot actually exist or remain in existence–disturbances cause it to pop or "wiggle" into another surface.

One area of minimal surface theory that has seen a lot of interest and results recently is the study of complete embedded minimal surfaces. Basically, these are minimal surfaces that are boundaryless (complete) and have no self-intersections (embedded). The plane, the catenoid, the helicoid, and Scherk's doubly periodic surface are examples of complete embedded minimal surfaces. However, the Enneper surface is not embedded, because it has self-intersections as its domain increases (see Exploration 2.11).

To begin to understand minimal surfaces, we need some tools from differential geometry, and these are discussed in Section 2. Section 3 uses material from the previous section to define a minimal surface and discuss some examples and properties of minimal surfaces. Section 4 brings in complex analysis to study minimal surfaces and introduces the Weierstrass representation formula to efficiently describe and study properties of minimal surfaces. These three sections are fundamental and should be read first. In Sections 5 - 7, we begin to explore ideas that lead to beginning research problems for students. Sections 5 and 6 are independent of one another. In Section 5 we present the Weierstrass representation in the form of the Gauss map and height differential, which is the basis for much of the current research about minimal surfaces in $\mathbb{R}^3$. Section 6 connects ideas about minimal surfaces with planar harmonic mappings in geometric function theory (i.e., the study of complex analysis from a geometric viewpoint). Section 7 is a new area of investigation that combines the ideas of the previous two sections and has several problems that can be explored by beginning students. In this chapter, there are four applets used and they can be accessed online at http://www.jimrolf.com/explorationsInComplexVariables/chapter2.html:

- *DiffGeomTool* is used to visualize and explore basic differential geometry concepts in $\mathbb{R}^3$ such as the graph of a parametrization of a surface, curves on a surface, tangent planes on a surface, and unit normals on a surface.
- *MinSurfTool* is used to visualize and explore minimal surfaces in $\mathbb{R}^3$ by using various forms of the Weierstrass representation.
- *ComplexTool* is used to plot the image of domains in $\mathbb{C}$ under complex-valued functions.
- *LinComboTool* is used to plot and explore the convex combination of complex-valued harmonic polygonal maps.

Each section of this chapter contains examples, exercises, and explorations that involve using the applets. You should do all of the exercises and explorations, many of which present surfaces and concepts that will be used later in the chapter (there are additional exercises at the end of the chapter). In addition, there are short projects and long projects that are suitable as research problems for undergraduates to explore. The goal of this chapter is not to give a comprehensive or step-by-step approach to this topic, but rather to get the reader engaged with the general notions, questions, and techniques of the area – but even more so, to encourage the reader to actively pose as well as pursue their own questions. To better understand the nature and purpose of this text, the reader should be sure to read the Introduction at the beginning of this book before proceeding.

2.2. Differential Geometry

Our goal is to develop the mathematics necessary to investigate minimal surfaces in $\mathbb{R}^3$. Such minimal surfaces minimize area locally and can be thought of as saddle surfaces. At each point, the bending upward in one direction is matched with the bending downward in the orthogonal direction. Such bending is known mathematically as curvature. So, to initially understand and investigate minimal surfaces, we must first understand curvature, a quantity that is measured using the tools of differential geometry. Differential geometry is a field of mathematics in which the ideas and techniques of calculus are applied to geometric shapes.

We will begin our discussion of differential geometry by looking at a surface in $\mathbb{R}^3$. Every point on a surface $M \subset \mathbb{R}^3$ can be designated by a point, $(x, y, z) \in \mathbb{R}^3$, but it can also be represented by two parameters. Let D be an open set in $\mathbb{R}^2$. Then the surface M can be represented by a function $\mathbf{x} : D \to \mathbb{R}^3$, where $\mathbf{x}(u, v) = (x_1(u, v), x_2(u, v), x_3(u, v))$ (that is, M is the image of $\mathbf{x}(D)$). We will require that $\mathbf{x}$ be differentiable. That is, each coordinate function $x_k(u, v)$ has continuous partial derivatives of all orders in D. Such a function or mapping is called a *parametrization.*

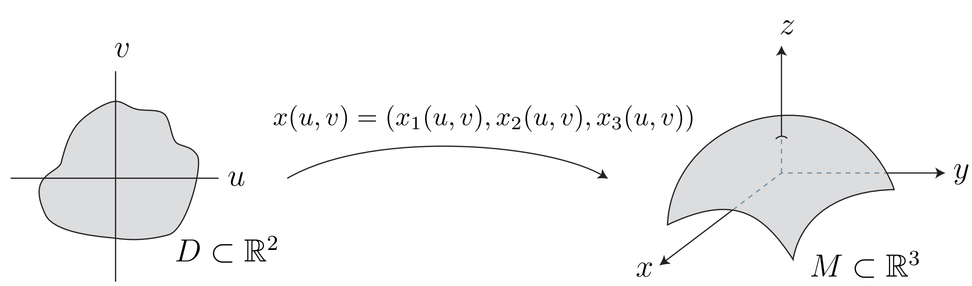


FIGURE 2.7. The parameterization of a surface

Let's consider two examples.

EXAMPLE 2.5. The Enneper surface is a minimal surface formed by bending a disk into a saddle surface. It can be parametrized by

$$\mathbf{x}(u, v) = \left(u - \frac{1}{3}u^3 + uv^2, v - \frac{1}{3}v^3 + u^2v, u^2 - v^2\right),$$

where u, v are in a disk of radius r. We can use the applet, *DiffGeomTool*, to graph this parametrization of the Enneper surface (see Figure 2.8). Open *DiffGeomTool* and enter the coordinate functions of the parametrization as

$$X(u, v) = u - 1/3 * u \wedge 3 + u * v \wedge 2$$
$$Y(u, v) = v - 1/3 * v \wedge 3 + u \wedge 2 * v$$
$$Z(u, v) = u \wedge 2 - v \wedge 2$$

into the appropriate boxes. In the gray part on the right side of the applet, click on `Circular grid` with radius min: 0.0, radius max: 1.0, theta min: 0.0, and theta max: 2$*$ pi. This is because we want our u, v values to be the unit disk. Then click the **Graph** button. To rotate the graph, place the cursor arrow on the image of the surface, and then click on and hold the left button on the mouse as you move the cursor. To increase the size of the image of the surface click on the left button on the mouse; to decrease the size, click on the right mouse button.

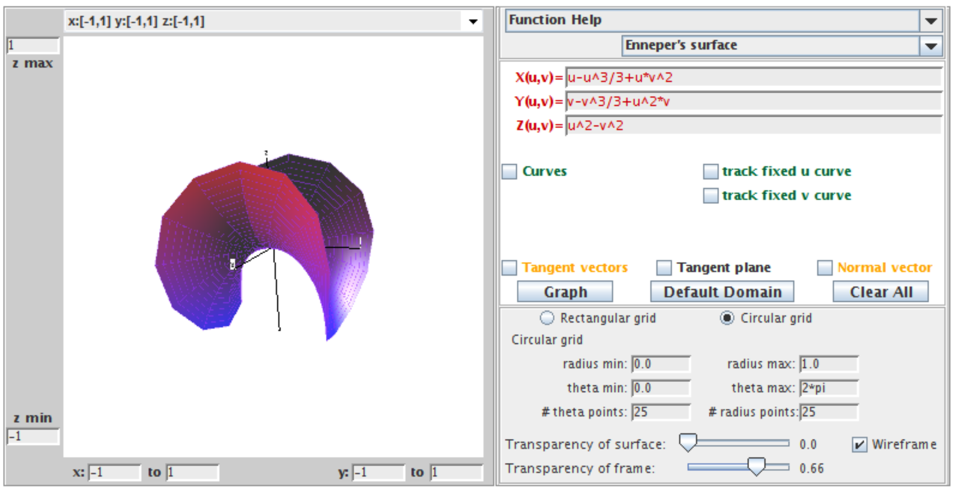


FIGURE 2.8. The Enneper surface.

EXAMPLE 2.6. If a heavy flexible cable is suspended between two points at the same height, then it takes the shape of a curve that can be described mathematically by the function $\alpha(t) = a\cosh(t/a)$. Such a curve is called a catenary from the Latin word that means "chain". In calculus, we discuss rotating a curve about a line to get a surface of revolution. We can apply this idea to the catenary curve to get a surface known as the catenoid. In particular, a *catenoid* is a surface that can be generated by rotating a catenary on its side about the x_3-axis (see Figure 2.9). A catenoid is also a minimal surface as we will verify in Section 2.3. How do we parametrize this catenoid? If we let $x_1 = a\cosh v$ $(-\infty < v < \infty)$ and $x_3 = av$, then $r(v) = (a\cosh v, av)$ is a parametrization of the catenary curve on its side in the x_1x_3-plane. Rotating a line about an axis is a circular motion, and a circle can be parametrized by $(\cos u, \sin u)$. So, we can parametrize this rotation of the catenary curve about the x_3-axis by multiplying

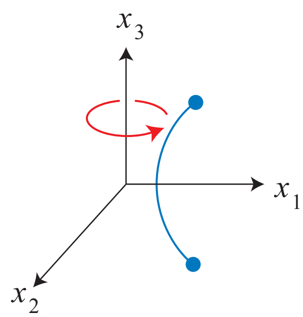


FIGURE 2.9. Creating a catenoid by rotating a catenary.

$a \cosh v$ by $\cos u$ for the x_1-coordinate function, and multiplying $a \cosh v$ by $\sin u$ for the x_2-coordinate function. Hence, we get the following parametrization for this catenoid surface:

$$\mathbf{x}(u, v) = \Big(a \cosh v \cos u, a \cosh v \sin u, av\Big).$$

Using *DiffGeomTool*, we can graph this parametrization of a catenoid with $a = 1$, clicking on `Rectangular grid`, and setting the boxes to 0 <= u <= 2 pi and -2pi/3 <= v <= 2pi/3 (see Figure 2.10). Note that $\cosh v, \cos u$, and $\sin u$ should be entered as $\cosh(v), \cos(u)$, and $\sin(u)$, respectively.

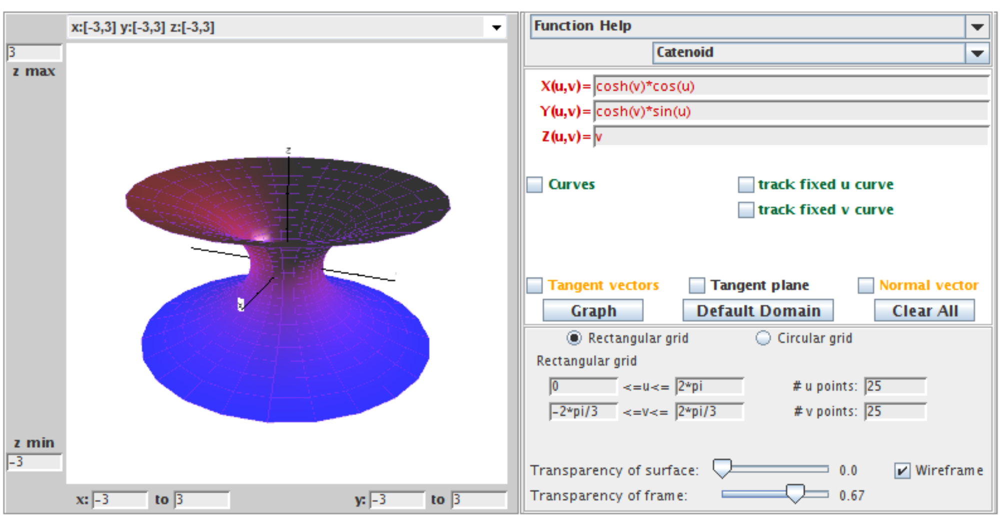


FIGURE 2.10. The catenoid.

Check out what happens if you change the u, v values. For example, try:

(a) pi <= u <= 2pi, -2pi/3 <= v <= 2pi/3;
(b) 0 <= u <= 2pi, 0 <= v <= 2pi/3;

(c) 0 <= u <= 2pi,   -pi/4 <= v <= pi/4;
(d) 0 <= u <= 2pi,   -pi <= v <= pi.

EXERCISE 2.7. A torus is a surface (but not a minimal surface) that can be formed by rotating a circle in the x_1x_3-plane about the x_3-axis. Let this be a circle of radius b and whose center is a distance of a from the origin.

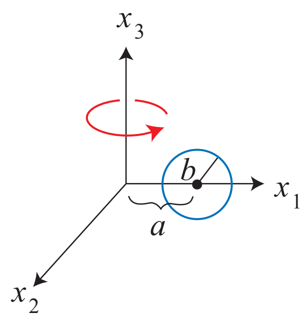


FIGURE 2.11. Creating a torus by rotating a circle.

Then the parametrization of this torus is

$$\mathbf{x}(u,v) = \Big((a+b\cos v)\cos u, (a+b\cos v)\sin u, b\sin v\Big),$$

where a, b are fixed, $0 < u < 2\pi$, and $0 < v < 2\pi$.

(a) Show how to derive this parametrization for a torus.
(b) Use *DiffGeomTool* to sketch the graph of this torus when $a = 3$ and $b = 2$; use `Rectangular grid` with $0 < u < 2\pi$, and $0 < v < 2\pi$.

Try it out!

In pre-calculus, we talk about a function of one variable $y = F(x)$ as one that satisfies the vertical line test. The graph of $F(x)$ is a 1-dimensional object, living in $\mathbb{R}^2$, created by plotting points of the form $(u, F(u))$ in the plane. Thus, it is parametrized by the map from $\mathbb{R}$ to $\mathbb{R}^2$ defined by $u \to (u, F(u))$. Analogously, we speak of a function of two variables $z = f(x, y)$, where the points (x, y) lie in a two-dimensional domain and f satisfies the vertical line test (here a line is vertical when it is parallel to the z-axis). The graph of $f(x, y)$ is a two-dimensional surface living in $\mathbb{R}^3$ with a height of $z = f(x, y)$ at a point (x, y) in its domain. An example of such a graph is the minimal surface known as Scherk's doubly periodic surface. It can be parametrized by

$$\mathbf{x}(u,v) = \left(u, v, \ln\left(\frac{\cos u}{\cos v}\right)\right).$$

EXERCISE 2.8.

(a) In this parametrization of Scherk's doubly periodic surface, what are the restrictions on the u and v values in the domain?

(b) Use *DiffGeomTool* and your answer from part (a) to sketch a graph of Scherk's doubly periodic surface with -0.48 pi <= u,v <= 0.48 pi.
(c) Scherk's doubly periodic surface is a particular example of the graph of a function. Now, let $f(x, y)$ be any function. Find a parametrization of the graph of f in general.

Try it out!

EXERCISE 2.9. Let r be a differentiable curve whose derivative does not vanish (i.e., $r'(v) \neq 0$ for all values v in the domain) and let r lie in some plane in $\mathbb{R}^3$. A *surface of revolution* is a surface that forms by rotating r about an axis in that plane such that the curve does not intersect the axis. The catenoid and torus are examples of this. For this exercise, let $r(v) = (f(v), 0, g(v))$ be such a curve in the x_1x_3-plane.

(a) Find a parametrization for the surface of revolution generated by rotating this curve about the x_3-axis.
(b) Check that your answer to part (a) matches the parametrizations of the catenoid and the torus given above.

Try it out!

EXPLORATION 2.10. Consider the torus $T_{a,b}$ whose parametrization is given in Exercise 2.7. Use *DiffGeomTool* to plot $T_{3,2}$ again. Describe what happens to the shape of the torus $T_{a,b}$ as as a gets smaller and b gets larger. Explain this in terms of how we derived the parametrization of the torus. [HINT: in *DiffGeomTool*, plot each of the following tori:

$$T_{2.7,2} \qquad T_{2.4,2} \qquad T_{2,2} \qquad T_{3,2.4} \qquad T_{3,2.7} \qquad T_{3,3}$$

What happens when $a < b$?]

Try it out!

EXPLORATION 2.11. As mentioned earlier, the Enneper surface is not embedded; that is, it has self-intersections. Use *DiffGeomTool* and the parametrization given in Example 2.5 to graph the Enneper surface with the domain being a disk of radius 1.

(a) What happens to the Enneper surface as the radius r of the disk increases?
(b) Estimate the largest value of r for which the Enneper surface has no self-intersections.
(c) Assuming that the intersection occurs on the x_3-axis, prove your result from part (b).

Try it out!

So far we have discussed how a function (i.e., a parametrization) models a surface. Our goal is to determine the bending or curvature of curves on a surface. To do this, we next will need to use the parametrization of a surface to discuss the concepts of a tangent plane and a normal vector at a point on the surface. Suppose $\mathbf{x}(u, v)$ is a parametrization of a surface $M \subset \mathbb{R}^3$. If we fix $v = v_0$ and let u vary, then $\mathbf{x}(u, v_0)$

depends on one parameter and is known as a *u-parameter curve*. Likewise, we can fix $u = u_0$ and let v vary to get a *v-parameter curve* $\mathbf{x}(u_0, v)$ (see Figure 2.12). Tangent vectors for the u-parameter and v-parameter curves are computed by differentiating the component functions of $\mathbf{x}$ with respect to u and v, respectively. That is, $\mathbf{x}_u$ and $\mathbf{x}_v$ are the tangent vectors defined by

$$\mathbf{x}_u = \left(\frac{\partial x_1}{\partial u}, \frac{\partial x_2}{\partial u}, \frac{\partial x_3}{\partial u}\right), \qquad \mathbf{x}_v = \left(\frac{\partial x_1}{\partial v}, \frac{\partial x_2}{\partial v}, \frac{\partial x_3}{\partial v}\right).$$

Note that for each point $p = x(u_0, v_0)$ on the surface, we get two corresponding vectors by plugging u_0 and v_0 in $\mathbf{x}_u$ and $\mathbf{x}_v$.

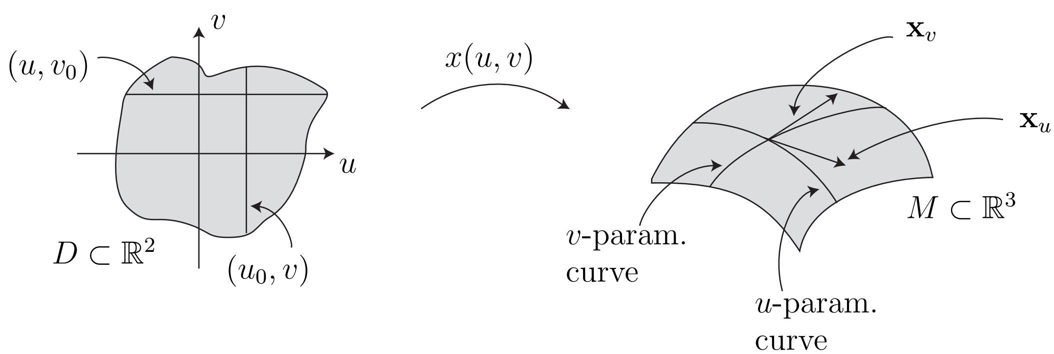


FIGURE 2.12. The u-parameter and v-parameter curves

Whenever we have a parametrization of a surface, we will require that $\mathbf{x}_u$ and $\mathbf{x}_v$ be linearly independent (i.e., not constant multiples of each other). Because of this, the span of $\mathbf{x}_u$ and $\mathbf{x}_v$ (i.e., the set of all vectors that can be written as a linear combination of $\mathbf{x}_u$, $\mathbf{x}_v$) forms a plane called the *tangent plane*. A tangent plane is an example of a vector space which are objects studied in linear algebra.

DEFINITION 2.12. The *tangent plane* of a surface M at a point p is

$$T_pM = \{\mathbf{v} \big| \mathbf{v} \text{ is tangent to } M \text{ at } p\}.$$

DEFINITION 2.13. The *unit normal* to a surface M at a point $p = \mathbf{x}(a, b)$ is

$$\mathbf{n}(a, b) = \left.\frac{\mathbf{x}_u \times \mathbf{x}_v}{\left|\mathbf{x}_u \times \mathbf{x}_v\right|}\right|_{(a,b)}.$$

Not every surface has a well-defined choice of a unit normal $\mathbf{n}$. Such surfaces are called *non-orientable*. An example of this is the Möbius strip. An orientable strip can be modeled by taking a strip of paper and taping the ends together. In this model, the strip has an outside and an inside–you <u>cannot</u> get from one side to the other without crossing over the edge. On the other hand, a Möbius strip is modeled by taking the

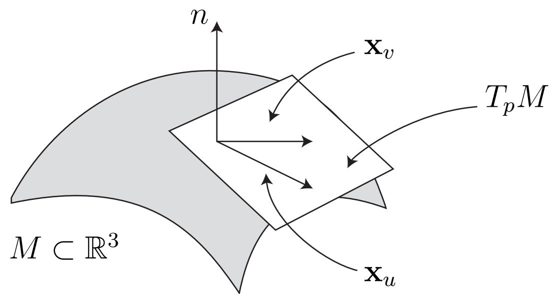


FIGURE 2.13. A tangent plane, T_pM, and unit normal vector, $\mathbf{n}$

same strip of paper, but twisting one end before taping the ends. This forms a strip in which you can get from one side to the other without crossing over the edge. This is the intuitive idea of a surface being non-orientable. The relationship between a Möbius strip and its unit normal $\mathbf{n}$ is further explored in Exercise 2.137 in the **Additional Exercises** at the end of this chapter. Note that the unit normal, $\mathbf{n}$, is orthogonal to the tangent plane at p (see Figure 2.13). Also, if the surface M is oriented, then geometrically there are two unit normals at each point $p \in M$ – an outward pointing normal and an inward pointing normal. However, the definition of $\mathbf{n}$ automatically chooses one of these normals.

EXAMPLE 2.14. Consider a torus parametrized by

$$\mathbf{x}(u,v) = \Big((3+2\cos v)\cos u, (3+2\cos v)\sin u, 2\sin v\Big),$$

where $0 < u, v < 2\pi$. For $v_0 = \frac{\pi}{3}$, the u-parameter curve is

$$\mathbf{x}\left(u, \frac{\pi}{3}\right) = (4\cos u, 4\sin u, \sqrt{3}).$$

For $u_0 = \frac{\pi}{2}$, the v-parameter curve is

$$\mathbf{x}\left(\frac{\pi}{2}, v\right) = (0, 3+2\cos v, 2\sin v).$$

Notice

$$\begin{aligned}\mathbf{x}_u(u,v) &= (-(3+2\cos v)\sin u, (3+2\cos v)\cos u, 0)\\ \mathbf{x}_v(u,v) &= (-2\sin v\cos u, -2\sin v\sin u, 2\cos v).\end{aligned}$$

Now the u-parameter curve, $\mathbf{x}\big(u, \frac{\pi}{3}\big)$, and the v-parameter curve, $\mathbf{x}\big(\frac{\pi}{2}, v\big)$, intersect on the torus at $p = \mathbf{x}\big(\frac{\pi}{2}, \frac{\pi}{3}\big)$. Then the tangent vectors to the u- and v-parameter curves at the point p are

$$\mathbf{x}_u\left(\frac{\pi}{2},\frac{\pi}{3}\right) = (-4,0,0)$$
$$\mathbf{x}_v\left(\frac{\pi}{2},\frac{\pi}{3}\right) = (0,-\sqrt{3},1).$$

These two vectors span the tangent plane, T_pM, at p. We compute that

$$\mathbf{x}_u\left(\frac{\pi}{2},\frac{\pi}{3}\right) \times \mathbf{x}_v\left(\frac{\pi}{2},\frac{\pi}{3}\right) = (-4,0,0)\times(0,-\sqrt{3},1) = (0,1,4\sqrt{3}).$$

Hence,

$$\mathbf{n}\left(\frac{\pi}{2},\frac{\pi}{3}\right) = \left(0,\frac{1}{7},\frac{4\sqrt{3}}{7}\right).$$

We can use *DiffGeomTool* to display this u-parameter curve, v-parameter curve, $\mathbf{x}_u$, $\mathbf{x}_v$, and $\mathbf{n}$. Enter the parametrization in this example for the torus. Then click `Curves`. A `Point location` box along with a `fixed u` and a `fixed v` boxes will appear. In the `Point location` box, enter pi/2 into the first box (i.e., the fixed u value) and pi/3 into the second box (i.e., the fixed v value). Make sure that the `fixed u` box is clicked but not the `fixed v` box. Then click the **Graph** button. The v-parameter curve will appear. If you now click the `track fixed u curve` box, a slider will appear. Moving the slider with the cursor will move the point along the v-parameter curve on the torus. Now, click on the `fixed u` box and then click the `Graph` button again. The u-parameter curve will appear. By clicking on the `track fixed v curve` box and moving the slider, the point along the v-parameter curve will move. Next, click on each of the following boxes separately followed by the **Graph** button: `Tangent vectors` box, `Tangent plane` box, and `Normal vector` box. This will cause these geometric objects to appear. You should convince yourself that the images of the vectors at $(u,v) = (\frac{\pi}{2},\frac{\pi}{3})$ match the computed values done earlier in Example 2.14.

EXERCISE 2.15. For a surface of revolution (see Exercise 2.9) parametrized by

$$\mathbf{x}(u,v) = \big(f(v)\cos u, f(v)\sin u, g(v)\big)$$

the u-parameter curves are called *parallels* and are the curves formed by horizontal slices, while the v-parameter curves are called *meridians* and are the curves formed by vertical slices. Describe the parallels and meridians for the catenoid in Example 2.6 and the torus in Exercise 2.7.

Try it out!

EXERCISE 2.16. Recall the parametrization of a catenoid

$$\mathbf{x}(u,v) = \Big(\cosh v\cos u, \cosh v\sin u, v\Big),$$

with $0 < u < 2\pi$ and $-\frac{2\pi}{3} < v < \frac{2\pi}{3}$.

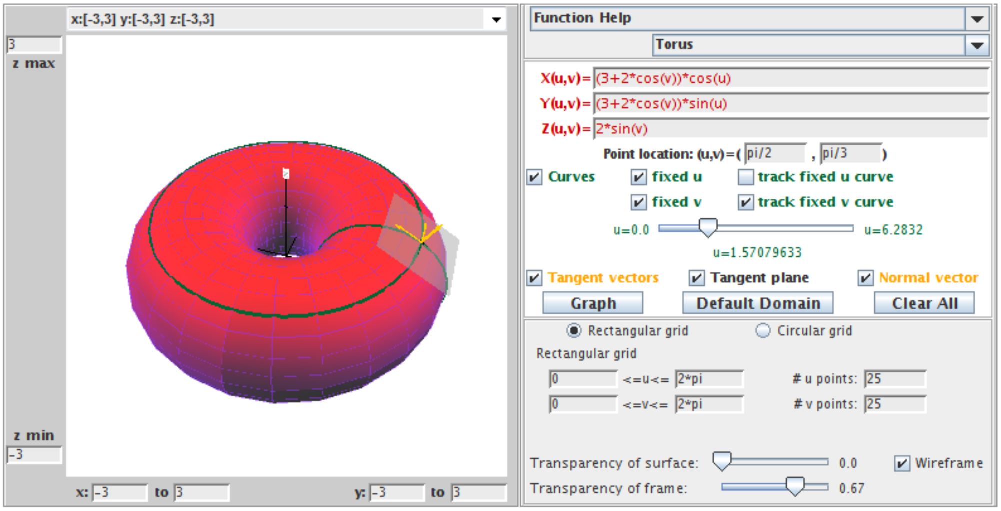


FIGURE 2.14. The torus with specific u, v-parameter curves, the tangent vectors, the tangent plane, and the normal vector.

(a) Use *DiffGeomTool* to sketch the u-parameter curve, $\mathbf{x}(u, 0)$, and the v-parameter curve, $\mathbf{x}(0, v)$ on the catenoid. Also, sketch the vectors $\mathbf{x}_u(0, 0)$, $\mathbf{x}_v(0, 0)$, and $\mathbf{n}(0, 0)$.
(b) Compute the vectors $\mathbf{x}_u(0, 0)$, $\mathbf{x}_v(0, 0)$, and $\mathbf{n}(0, 0)$.

Try it out!

EXERCISE 2.17. Recall the parametrization of Scherk's doubly periodic surface

$$\mathbf{x}(u, v) = \left(u, v, \ln\left(\frac{\cos u}{\cos v}\right)\right),$$

with $-0.48\pi < u, v < 0.48\pi$.

(a) Use *DiffGeomTool* to sketch the u-parameter curve, $\mathbf{x}\left(u, \frac{\pi}{4}\right)$, and the v-parameter curve, $\mathbf{x}\left(\frac{\pi}{4}, v\right)$ on Scherk's doubly periodic surface (make sure you use $-0.48\pi < u, v < 0.48\pi$). Also, sketch the vectors $\mathbf{x}_u\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$, $\mathbf{x}_v\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$, and $\mathbf{n}\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$. Note that you can slide these vectors by clicking the `track fixed u curve` box. This collection of vectors, $(\mathbf{x}_u, \mathbf{x}_v, \mathbf{n})$, is known as a *moving frame* or *Frenet frame* of a curve. The way these vectors vary in $\mathbb{R}^3$ as the frame moves along the curve describes how the curve twists and turns in $\mathbb{R}^3$. For more details, see [**18**] or [**21**].
(b) Compute the vectors $\mathbf{x}_u\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$, $\mathbf{x}_v\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$, and $\mathbf{n}\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$.

Try it out!

We will use the normal vector $\mathbf{n}$ to define the curvature of a curve on a surface. As we mentioned in the introduction, a minimal surface has very particular behavior with respect to curvature: at each point on the surface, the bending upward in one direction is matched with the bending downward in the orthogonal direction. Hence, we will use this definition of curvature in our definition of a minimal surface.

Notice that any plane containing the normal $\mathbf{n}$ will intersect the surface M in a curve, α. For each curve α, we can compute its curvature, which measures how fast the curve pulls away from the tangent line at p. So let's now consider some ideas about the curvature of a curve. Any curve in $\mathbb{R}^3$ can be parametrized by a function of one variable, say $\alpha(t)$, where $\alpha : [a, b] \to \mathbb{R}^3$. However, this parametrization is not unique.

EXERCISE 2.18. Find two different parametrizations of the unit circle in the x_1x_2-plane.

Try it out!

This lack of uniqueness can cause difficulties in exploring the concept of curvature. To eliminate these difficulties we will standardize our parametrization by requiring it to be a unit speed curve.

DEFINITION 2.19. A curve α is a *unit speed curve* if $\left|\alpha'(t)\right| = 1$.

If our parametrization of our regular curve $\alpha(t)$ is not of unit speed, we can always reparametrize it by arclength to form a unit speed curve $\alpha(s)$ (you probably saw this in calculus 3 or vector calculus, and you may want to review how to reparametrize a curve by arclength). Because of this, we will assume that the curves we will be discussing are unit speed curves $\alpha(s)$. This assumption means that we are only interested in the geometric shape of a regular curve since reparametrizing does not change its shape.

Given a curve α, we want to discuss its curvature (or bending). We quantify the amount that the curve bends at each point p by measuring how rapidly the curve pulls away from the tangent line at p. In other words, we measure the amount of bending by measuring the rate of change of the angle θ that neighboring tangents make with the tangent at p. Thus, we are interested in the rate of change of the tangent vector (i.e., the value of the second derivative).

DEFINITION 2.20. The *curvature* of the unit speed curve α at s is $\left|\alpha''(s)\right|$.

EXAMPLE 2.21. Consider a torus parametrized by

$$\mathbf{x}(u, v) = \Big((3 + 2\cos v)\cos u, (3 + 2\cos v)\sin u, 2\sin v\Big),$$

where $0 < u, v < 2\pi$. Let's compute the curvature for the u-parameter curves and the v-parameter curves. All the v-parameter curves (or meridians) are the curves formed by vertical slices of the torus, and hence are circles of radius $b = 2$. To compute the curvature of these v-parameter curves, we start with the parametrization of these curves

$$\mathbf{x}(v) = \mathbf{x}(u_0, v) = \Big((3 + 2\cos v)\cos u_0, (3 + 2\cos v)\sin u_0, 2\sin v\Big),$$

where u_0 is a fixed value. We next need to reparametrize $\mathbf{x}(v)$ so that it is a unit speed curve. Differentiating $\mathbf{x}(v)$ with respect to v we get

$$\mathbf{x}'(v) = \big(- 2\sin v \cos u_0, -2\sin v \sin u_0, 2\cos v\big).$$

Thus, $|\mathbf{x}'(v)| = 2$. To make $\mathbf{x}'(v)$ into a unit speed curve, we replace v with $\frac{s}{2}$. So, our reparametrized curve is

$$\mathbf{x}(s) = \left(\left(3 + 2\cos\left(\frac{s}{2}\right)\right)\cos u_0, \left(3 + 2\cos\left(\frac{s}{2}\right)\right)\sin u_0, 2\sin\left(\frac{s}{2}\right)\right).$$

Then we compute

$$\mathbf{x}'(s) = \left(-\sin\left(\frac{s}{2}\right)\cos u_0, -\sin\left(\frac{s}{2}\right)\sin u_0, \cos\left(\frac{s}{2}\right)\right)$$
$$\mathbf{x}''(s) = \left(-\frac{1}{2}\cos\left(\frac{s}{2}\right)\cos u_0, -\frac{1}{2}\cos\left(\frac{s}{2}\right)\sin u_0, -\frac{1}{2}\sin\left(\frac{s}{2}\right)\right).$$

Hence, the curvature of the v-parameter curves is

$$\big|\mathbf{x}''(s)\big| = \frac{1}{2}.$$

The u-parameter curves (or parallels) are the curves formed by horizontal slices of the torus, and so are circles of radius $3 + 2\cos v_0$, where $v_0 \in (0, 2\pi)$ is fixed; note that these radii vary between 1 and 5. These curves are parametrized by

$$\mathbf{x}(u) = \mathbf{x}(u, v_0) = \Big((3 + 2\cos v_0)\cos u, (3 + 2\cos v_0)\sin u, 2\sin v_0\Big),$$

which are reparametrized to the unit speed curve

$$\mathbf{x}(s) = \left((3 + 2\cos v_0)\cos\left(\frac{s}{3 + 2\cos v_0}\right), (3 + 2\cos v_0)\sin\left(\frac{s}{3 + 2\cos v_0}\right), 2\sin v_0\right).$$

Finally, computing the curvature of these u-parameter curves yields

$$\big|\mathbf{x}''(s)\big| = \frac{1}{3 + 2\cos v_0}.$$

So, the curvature of the these curves varies between $\frac{1}{5}$ and 1.

Exercise 2.22. Verify that replacing v with $\frac{s}{2}$ in $\mathbf{x}(u, v)$ from the previous example gives a unit speed curve.

Try it out!

Exercise 2.23. Compute the curvatures of the meridians and parallels of the catenoid parametrized by

$$\mathbf{x}(u, v) = \Big(a\cosh v \cos u, a\cosh v \sin u, av\Big).$$

Try it out!

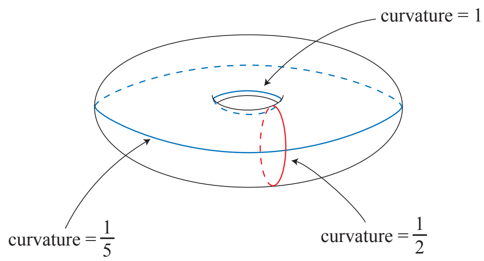


FIGURE 2.15. The curvature of the meridians and parallels on a torus.

EXERCISE 2.24. The curve parametrized by $\alpha(t) = (a\cos t, a\sin t, bt)$ is known as a *helix* which is a spiral that rises with a pitch of $2\pi b$ on the cylinder $x^2 + y^2 = a^2$.

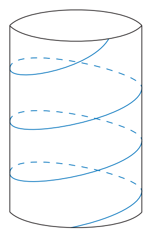


FIGURE 2.16. A helix in a cylinder.

We can create a surface by connecting a line from the axis $(0, 0, bt)$ through the helix $(a\cos t, a\sin t, bt)$ at each height bt. This ruled surface is a minimal surface known as a *helicoid*. All minimal surfaces including the helicoid can be parametrized in several ways. For our purposes, we will use the following parametrization of the helicoid:

$$\mathbf{x}(u, v) = \Big(a\sinh v\cos u, a\sinh v\sin u, au\Big).$$

(a) Compute the curvatures of the u-parameter curves and v-parameter curves of this helicoid (note: making the v-parameter curve into a unit speed curve is not easy, so in doing this computation you may need to be creative).

(b) Use *DiffGeomTool* to graph this helicoid with $a = 1$ (see Figure 2.17).

Try it out!

EXERCISE 2.25. From the results of Example 2.21 you may have conjectured that the curvature of a circle of radius r is $\dfrac{1}{r}$. This conjecture is correct. Prove this.

Try it out!

Now let's return to surfaces. Suppose we have a curve $\sigma(s)$ on a surface M. We can determine the unit tangent vector, $\mathbf{w}$ of σ at $p \in M$ and the unit normal, $\mathbf{n}$ of M at $p \in M$. Note that $\mathbf{w} \times \mathbf{n}$ forms a plane $\mathcal{P}$ that intersects M creating a curve $\alpha(s)$.

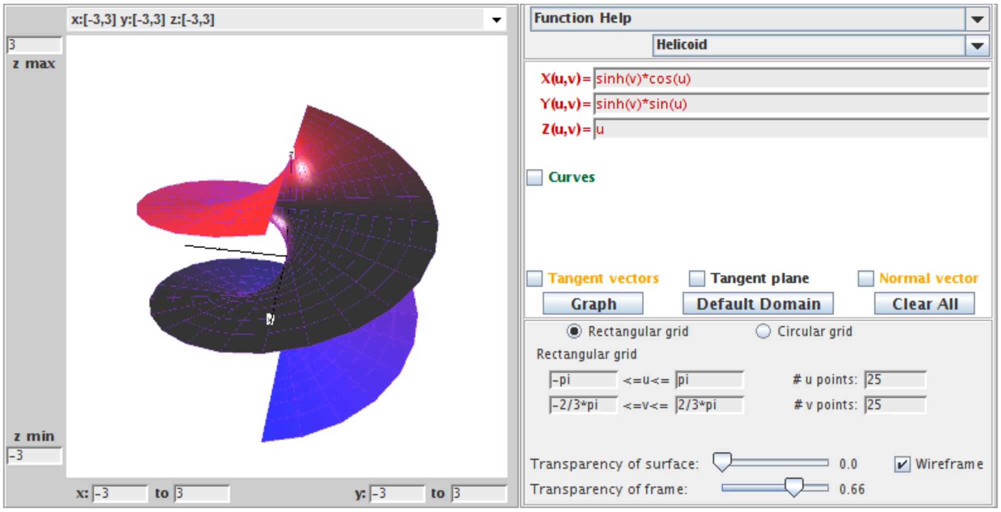


FIGURE 2.17. Helicoid.

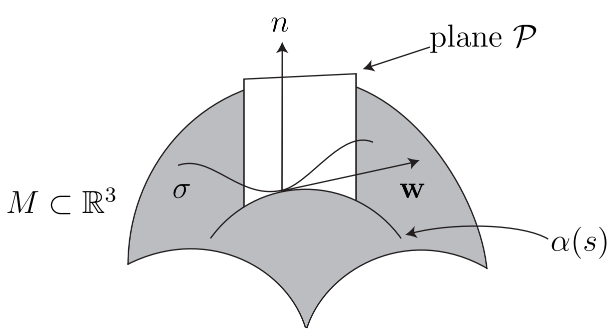


FIGURE 2.18. The normal curvature

DEFINITION 2.26. The *normal curvature* in the $\mathbf{w}$ direction is

$$k(\mathbf{w}) = \alpha'' \cdot \mathbf{n}.$$

Recall $\alpha'' \cdot \mathbf{n} = \left|\alpha''\right|\left|\mathbf{n}\right| \cos\theta$, where θ is the angle between $\mathbf{n}$ and α''. Hence $\alpha'' \cdot \mathbf{n}$ is the projection of α'' onto the unit normal (hence, the name *normal* curvature). Intuitively, the normal curvature measures how much the surface bends towards $\mathbf{n}$ as you travel in the direction of the tangent vector $\mathbf{w}$ starting at point p. As we rotate the plane about the normal $\mathbf{n}$, we will get a set of curves on the surface each of which has a value for its curvature. Let k_1 and k_2 be the maximum and minimum curvature values at p, respectively. The directions in which the normal curvature attains its absolute maximum and absolute minimum values are known as the *principal directions*.

DEFINITION 2.27. The *mean curvature* (i.e., average curvature) of a surface M at p is

$$H = \frac{k_1 + k_2}{2}.$$

It turns out that k_1 and k_2 come from two orthogonal tangent vectors. The mean curvature depends upon the point $p \in M$. However, it can be shown that H does not change if we choose any two orthogonal vectors and use their curvature values to compute H at p. Also, we will use the convention that if the principal curve with normal curvature k_j is bending toward the unit normal $\mathbf{n}$, then $k_j > 0$ and if it is bending away from $\mathbf{n}$, then $k_j < 0$.

EXAMPLE 2.28. At any point on a sphere of radius a, all the curves α are circles of radius a and hence have the same curvature value which can be computed to be $1/a$. Since these curves are bending away from $\mathbf{n}$, $k_1 = -1/a = k_2$. So the mean curvature is $-1/a$.

EXERCISE 2.29. Determine the mean curvature at all points on the cylinder parametrized by $\mathbf{x}(u, v) = (a \cos u, a \sin u, bv)$.

Try it out!

EXERCISE 2.30. Determine if there are points on the torus $\mathbf{x}(u, v) = \big((a+b\cos v)\cos u, (a+b\cos v)\sin u, b \sin v\big)$ where $H > 0$, $H = 0$, and $H < 0$.

Try it out!

In the next section of this chapter we will define a minimal surface in terms of mean curvature. However, using our current definition it is tedious to compute the mean curvature of a surface because we have to find the principle curvatures individually for every point on the surface. Instead, we would like to have an explicit expression to compute mean curvature at every point p on the surface simply by plugging p into the formula (or, more precisely, plugging in each point (u, v) in the domain of the parametrization of the surface). Fortunately, there is a more useful formula for mean curvature using the coefficients of the first and second fundamental forms for a surface. Recall that α is a unit speed curve. Hence

$$\begin{aligned} 1 =& |\alpha'|^2 = \alpha' \cdot \alpha' \\ =& (\mathbf{x}_u \, du + \mathbf{x}_v \, dv) \cdot (\mathbf{x}_u \, du + \mathbf{x}_v \, dv) \\ =& \mathbf{x}_u \cdot \mathbf{x}_u \, du^2 + 2\mathbf{x}_u \cdot \mathbf{x}_v \, dudv + \mathbf{x}_v \cdot \mathbf{x}_v \, dv^2 \\ =& E \, du^2 + 2F \, dudv + G \, dv^2. \end{aligned} \tag{2}$$

The terms $E = \mathbf{x}_u \cdot \mathbf{x}_u$, $F = \mathbf{x}_u \cdot \mathbf{x}_v$, and $G = \mathbf{x}_v \cdot \mathbf{x}_v$ are known as the *coefficients of the first fundamental form*. These describe how lengths on a surface are distorted as compared to their usual measurements in $\mathbb{R}^3$.

Next, recall $k(\mathbf{w}) = \alpha'' \cdot \mathbf{n}$. Note that $\alpha' \cdot \mathbf{n} = 0$, and so $(\alpha' \cdot \mathbf{n})' = 0$, which implies $\alpha'' \cdot \mathbf{n} + \alpha' \cdot \mathbf{n}' = 0$, and thus $\alpha'' \cdot \mathbf{n} = -\alpha' \cdot \mathbf{n}'$. Similarly, $-\mathbf{x}_u \cdot \mathbf{n}_u = \mathbf{x}_{uu} \cdot \mathbf{n}$. So

$$\begin{aligned} k(\mathbf{w}) &= -\,\alpha' \cdot \mathbf{n}' \\ &= -\,(\mathbf{x}_u\, du + \mathbf{x}_v\, dv) \cdot (\mathbf{n}_u\, du + \mathbf{n}_v\, dv) \\ &= -\,\mathbf{x}_u \cdot \mathbf{n}_u\, du^2 - (\mathbf{x}_u \cdot \mathbf{n}_v + \mathbf{x}_v \cdot \mathbf{n}_u)\, dudv - \mathbf{x}_v \cdot \mathbf{n}_v\, dv^2 \\ &= \mathbf{x}_{uu} \cdot \mathbf{n}\, du^2 + 2\mathbf{x}_{uv} \cdot \mathbf{n}\, dudv + \mathbf{x}_{vv} \cdot \mathbf{n}\, dv^2 \\ &= e\, du^2 + 2f\, dudv + g\, dv^2. \end{aligned}$$

The terms $e = \mathbf{x}_{uu} \cdot \mathbf{n}$, $f = \mathbf{x}_{uv} \cdot \mathbf{n}$, and $g = \mathbf{x}_{vv} \cdot \mathbf{n}$ are called the *coefficients of the second fundamental form*. These describe how much the surface bends away from the tangent plane.

EXAMPLE 2.31. Recall that a catenoid can be parametrized by

$$\mathbf{x}(u, v) = \Big(a \cosh v \cos u, a \cosh v \sin u, av\Big).$$

Using this parametrization, we compute that

$$\begin{aligned} \mathbf{x_u} &= (-a \cosh v \sin u, a \cosh v \cos u, 0) \\ \mathbf{x_v} &= (a \sinh v \cos u, a \sinh v \sin u, a). \end{aligned}$$

So, the coefficients of the first fundamental form are:

$$\begin{aligned} E &= \mathbf{x}_u \cdot \mathbf{x}_u = a^2 \cosh^2 v; \\ F &= \mathbf{x}_u \cdot \mathbf{x}_v = 0; \\ G &= \mathbf{x}_v \cdot \mathbf{x}_v = a^2 \cosh^2 v. \end{aligned}$$

What do these values for E, F, and G tell us? Let $(u_0, v_0) \in D$ be a point in the domain and let's take a small square with a vertex at this point. Because $\mathbf{x}_u \cdot \mathbf{x}_v = F = 0$, we know that the orthogonal lines from the u-parameter curve and the v-parameter curve will remain orthogonal on the catenoid. That is, small squares will be mapped to small rectangles. Next, because $E = G$, adjacent sides of the image rectangle will have the same length. So, in fact, small squares in the domain D will be mapped to small squares on the catenoid. Now suppose for simplicity sake that $a = 1$. Then $E = G = \cosh^2 v$. When $v = 0$, $E = G = 1$ and as v gets farther away from 0, E and G get larger. This means that a small square containing the u-parameter curve $v = 0$ will get mapped to a small square of the same size on the catenoid. But as v gets farther away from 0, the size of the side lengths of the image square will increase by a factor of $\cosh^2 v$. This can be seen in Figure 2.19 where we have used the `Transparency of surface` and the `Transparency of the frame` sliders to make the u- and v-parameter curves more distinct (note that the u-parameter curve with $v = 0$ gets mapped to a parallel on the neck of the catenoid, and the u-parameter curve with $v = \frac{2\pi}{3}$ gets mapped to the edge of the catenoid, as displayed in the figure).

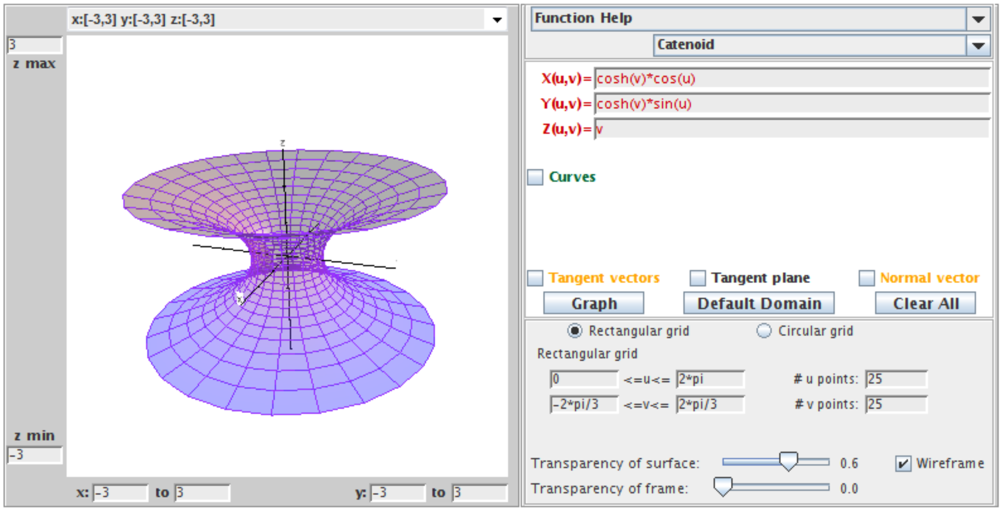


FIGURE 2.19. The catenoid.

In order to compute the coefficients of the second fundamental form, we need to compute $\mathbf{n} = \dfrac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}$. Now,

$$\mathbf{x}_u \times \mathbf{x}_v = (a^2 \cosh v \cos u, a^2 \cosh v \sin u, -a^2 \cosh v \sinh v),$$

and so

$$|\mathbf{x}_u \times \mathbf{x}_v| = a^2 \cosh^2 v.$$

Hence

$$\mathbf{n} = \left(\frac{\cos u}{\cosh v}, \frac{\sin u}{\cosh v}, -\frac{\sinh v}{\cosh v} \right).$$

Also, we can compute that

$$\begin{aligned} \mathbf{x}_{uu} =&(-a \cosh v \cos u, -a \cosh v \sin u, 0) \\ \mathbf{x}_{uv} =&(-a \sinh v \sin u, -a \sinh v \cos u, 0) \\ \mathbf{x}_{vv} =&(a \cosh v \cos u, a \cosh v \sin u, 0) \end{aligned}$$

Therefore, the coefficients of the second fundamental form are:

$$\begin{aligned} e =&\mathbf{n} \cdot \mathbf{x}_{uu} = -a; \\ f =&\mathbf{n} \cdot \mathbf{x}_{uv} = 0; \\ g =&\mathbf{n} \cdot \mathbf{x}_{vv} = a. \end{aligned}$$

What do these values for e, f, and g tell us? Again, let $(u_0, v_0) \in D$ be a point in the domain, and let $p \in M$ be the image of (u_0, v_0) on the surface. Then at p, the vectors $\mathbf{x}_u$ and $\mathbf{x}_v$ create the tangent plane T_pM and the unit normal $\mathbf{n}$. For this catenoid, the u-parameter curve is bending away from $\mathbf{n}$ while the v-parameter curve is bending toward $\mathbf{n}$. However, both curves are bending the same amount away from the tangent plane. This is represented by the fact that $e = -g$ in this example.

EXERCISE 2.32. A torus has the parametrization

$$\mathbf{x}(u,v) = \Big((a + b\cos v)\cos u, (a + b\cos v)\sin u, b\sin v\Big).$$

(a) Compute the coefficients of the first and the second fundamental forms.
(b) Open *DiffGeomTool* and enter this parametrization for the torus with $a = 2$ and $b = 1$. Use the `Rectangular grid` with 0 <= u <= 2* pi and 0 <= v <= 2* pi. And set the # u points: to 20 and the # v points: to 20. Describe how the results from part (a) match with the image of the torus in *DiffGeomTool*. Use the `Transparency of surface` and the `Transparency of the frame` sliders and click off and on the `Wireframe` button to make the wireframe clearer.

Try it out!

EXERCISE 2.33. Compute the coefficients of the first and the second fundamental forms for Scherk's doubly periodic surface parametrized by

$$\mathbf{x}(u,v) = \left(u, v, \ln\left(\frac{\cos u}{\cos v}\right)\right).$$

Try it out!

Now we want to express the mean curvature H in terms of these coefficients of the first and second fundamental forms. In particular, we will show that

$$H = \frac{Eg + Ge - 2Ff}{2(EG - F^2)}.$$

PROOF. There is an elegant way to derive this formula. This approach requires some concepts that are interesting but beyond the scope of what we will need. So instead we will use a straightforward calculation that is in Oprea ([**22**], pp. 40-42). Although this calculation does not give much insight into the formula, it does provide a straightforward proof of this important formula. For a discussion involving the more advanced approach, see [**3**] or [**18**].

Let $\mathbf{w}_1, \mathbf{w}_2$ be any two perpendicular unit vectors and k_1, k_2 be their normal curvatures using the curves $\alpha_1(s) = (u_1(s), v_1(s))$ and $\alpha_2(s) = (u_2(s), v_2(s))$. Also, let $p_1 = du_1 + idu_2$ and $p_2 = dv_1 + idv_2$. Then using the second fundamental form to compute k_1 and k_2, we have

$$\begin{aligned} 2H = k_1 + k_2 &= e(du_1^2 + du_2^2) + 2f(du_1dv_1 + du_2dv_2) + g(dv_1^2 + dv_2^2) \\ &= e(p_1\,\overline{p_1}) + f(p_1\,\overline{p_2} + \overline{p_1}p_2) + g(p_2\,\overline{p_2}). \end{aligned}$$

We want to further simplify this so that p_1 and p_2 do not appear in the expression. Recall eq (2):

$$1 = E\,du^2 + 2F\,dudv + G\,dv^2.$$

Also, note that $Edu_1du_2 + F(du_1dv_2 + du_2dv_1) + Gdv_1dv_2 = 0$, because $\mathbf{w}_1, \mathbf{w}_2$ are perpendicular and so $\mathbf{w}_1 \cdot \mathbf{w}_2 \alpha_1'(s) \cdot \alpha_2'(s) = 0$.

Now consider

$$\begin{aligned}
Ep_1^2 + 2Fp_1p_2 + Gp_2^2 &= E\big[du_1^2 - du_2^2 + i2du_1du_2\big] + 2F\big[du_1dv_1 - du_2dv_2 + i(du_1dv_2 + du_2dv_1)\big] \\
&\qquad + G\big[dv_1^2 - dv_2^2 + i2dv_1dv_2\big] \\
&= 2i\big[Edu_1du_2 + F(du_1dv_2 + du_2dv_1) + Gdv_1dv_2\big] \\
&\qquad + \big[Edu_1^2 + 2Fdu_1dv_1 + Gdv_1^2\big] - \big[Edu_2^2 + 2Fdu_1dv_2 + Gdv_2^2\big] \\
&= 0 + 1 - 1 \\
&= 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
p_1 =& \frac{-2Fp_2 \pm \sqrt{4F^2p_2^2 - 4EGP_2^2}}{2E} = \left(-\frac{F}{E} \pm i\frac{\sqrt{EG - F^2}}{E}\right)p_2 \\
\overline{p_1} =& \left(-\frac{F}{E} \mp i\frac{\sqrt{EG - F^2}}{E}\right)\overline{p_2}.
\end{aligned}$$

And so

$$p_1\,\overline{p_1} = \left(\frac{F^2}{E^2} + EG - F^2E\right)p_2\,\overline{p_2} = \frac{G}{E}p_2\,\overline{p_2}, \text{ and} \tag{3}$$

$$p_1\,\overline{p_2} + \overline{p_1}p_2 = -\frac{2F}{E}p_2\,\overline{p_2}. \tag{4}$$

Now we have

$$\begin{aligned}
2H = k_1 + k_2 &= e(p_1\,\overline{p_1}) + f(p_1\,\overline{p_2} + \overline{p_1}p_2) + g(p_2\,\overline{p_2}) \\
&= \left[e\frac{G}{E} + f\left(\frac{-2F}{E}\right) + g\right]p_2\,\overline{p_2}.
\end{aligned}$$

We just need to get rid of $p_2\,\overline{p_2}$. Again using eq (2), we have

$$\begin{aligned}
Ep_1\,\overline{p_1} &+ F(p_1\,\overline{p_2} + \overline{p_1}p_2) + Gp_2\,\overline{p_2} \\
&= E(du_1^2 + du_2^2) + 2F(du_1dv_1 + du_2dv_2) + G(dv_1^2 + dv_2^2) \\
&= 1 + 1 = 2.
\end{aligned}$$

Using eqs (3) and (4),we derive

$$\begin{aligned}
2 &= E\left(\frac{G}{E}p_2\,\overline{p_2}\right) + 2F\left(\frac{-2F}{E}p_2\,\overline{p_2}\right) + Gp_2\,\overline{p_2} \\
\Rightarrow 2 &= \left[2G - \frac{2F^2}{E}\right]p_2\,\overline{p_2} \\
\Rightarrow p_2\,\overline{p_2} &= \frac{E}{EG - F^2}
\end{aligned}$$

Therefore,

$$H = \frac{1}{2}\left[e\frac{G}{E} + f\left(\frac{-2F}{E}\right) + g\right]p_2\,\overline{p_2} = \frac{Eg + eG - 2Ff}{2(EG - F^2)}.$$

□

2.3. Minimal Surfaces

Now that we have a foundation of some essential ideas from differential geometry, we can begin to explore minimal surfaces. Earlier we mentioned that minimal surfaces can be thought of as saddle surfaces. That is, at each point the bending upward in one direction is matched with the bending downward in the orthogonal direction. This picture can be described mathematically with the following definition.

DEFINITION 2.34. A *minimal surface* is a surface M with mean curvature $H = 0$ at all points $p \in M$.

Make sure that you understand how this definition fits with the picture of a surface bending upward in one direction while also bending downward in the orthogonal direction. At this point we can use the results from the previous section. First, we can use the formula

$$H = \frac{Eg + Ge - 2Ff}{2(EG - F^2)}. \tag{5}$$

to show that a surface with a specific parametrization is minimal.

EXAMPLE 2.35. We will use eq (5) to show that the catenoid is a minimal surface. Recall that a catenoid can be parametrized by

$$\mathbf{x}(u, v) = \Big(a\cosh v\cos u, a\cosh v\sin u, av\Big).$$

From Example 2.31

$$\begin{aligned} E =& \mathbf{x}_u \cdot \mathbf{x}_u = a^2\cosh^2 v, \\ F =& \mathbf{x}_u \cdot \mathbf{x}_v = 0, \\ G =& \mathbf{x}_v \cdot \mathbf{x}_v = a^2\cosh^2 v, \end{aligned}$$

and

$$\begin{aligned} e =& \mathbf{n} \cdot \mathbf{x}_{uu} = -a, \\ f =& \mathbf{n} \cdot \mathbf{x}_{uv} = 0, \\ g =& \mathbf{n} \cdot \mathbf{x}_{vv} = a. \end{aligned}$$

Hence

$$H = \frac{eG - 2fF + Eg}{2(EG - F^2)} = 0.$$

And so the catenoid is a minimal surface.

EXERCISE 2.36. Using the parametrization for the helicoid

$$\mathbf{x}(u,v) = (a\sinh v\cos u, a\sinh v\sin u, au),$$

prove that the helicoid is a minimal surface. Using *DiffGeomTool* we display the graph of this helicoid when $a = 1$.

Try it out!

EXERCISE 2.37. Using the parametrization for the torus

$$\mathbf{x}(u,v) = \big((a+b\cos v)\cos u, (a+b\cos v)\sin u, b\sin v\big),$$

prove that it is not a minimal surface.

Try it out!

EXERCISE 2.38. Suppose a surface M is the graph of a function $f(x,y)$ of two variables (see the paragraph before Exercise 2.8). Then M can be parametrized by

$$\mathbf{x}(x,y) = \big(x, y, f(x,y)\big),$$

where its domain is formed by the projection of M onto the xy-plane.

(a) Compute the coefficients of the first and second fundamental forms for M.

(b) A *minimal graph* is a minimal surface that is a graph of a function. Prove

(6) M is a minimal graph if and only if $f_{xx}\left(1+f_y^2\right) - 2f_xf_yf_{xy} + f_{yy}\left(1+f_x^2\right) = 0.$

Try it out!

In the paragraph before Exercise 2.8, we stated that Scherk's doubly periodic surface is a minimal graph. We will now use eq (6) to prove that. Applying this equation is usually not easy, because solving explicitly for f can be complicated. However, one case in which we can do this is when f can be separated into two functions, each of which is dependent upon only one variable. In particular, suppose $f(x,y) = g(x) + h(y)$. Then the minimal surface equation becomes:

$$g''(x)[1+(h'(y))^2] + h''(y)[1+(g'(x))^2] = 0.$$

This is a separable differential equation and hence can be solved. To do so, separate all the terms with the x variables from those with the y variables by putting them on opposite sides. This yields:

$$-\frac{1+(g'(x))^2}{g''(x)} = \frac{1+(h'(y))^2}{h''(y)}. \tag{7}$$

What does this mean? If we fix y, the right side remains constant even if we change x in the left side. The same is true if we fix x and vary y. The only way such a situation can occur is if both sides are constant. So we have:

$$-\frac{1+(g'(x))^2}{g''(x)} = k \qquad \Longrightarrow \qquad 1+(g'(x))^2 = -kg''(x).$$

To solve this, let $\phi(x) = g'(x)$. Then $\frac{d\phi}{dx} = g''(x)$ and so

$$\int dx = -k \int \frac{d\phi}{1+\phi^2}$$
$$\implies \quad x = -k \arctan \phi + C$$
$$\implies \quad \phi = -\tan\left(\frac{x+C}{k}\right).$$

For convenience, let $C = 0$ and $k = 1$. Since $\phi = g'$, we can integrate again to get:

$$g(x) = \ln[\cos x].$$

Completing the same calculations for the y-side of eq (7) yields:

$$h(y) = -\ln[\cos y].$$

Hence

$$f(x,y) = g(x) + h(y) = \ln\left[\frac{\cos x}{\cos y}\right]$$

which is an equation for Scherk's doubly periodic surface. Using *DiffGeomTool* we display the graph of Scherk's doubly periodic surface (see Figure 2.20).

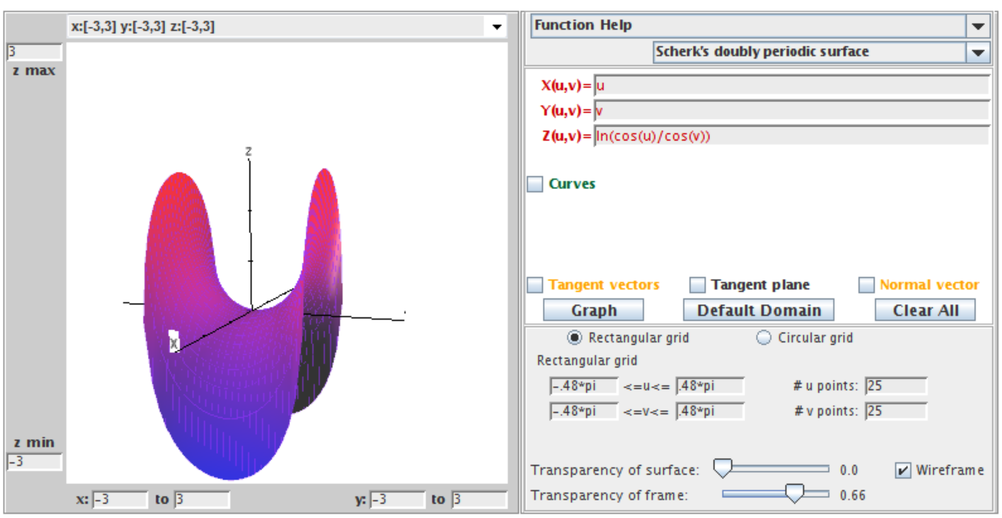


FIGURE 2.20. Scherk's doubly periodic surface.

Notice that $-\frac{\pi}{2} < x, y < \frac{\pi}{2}$ and so this surface is defined over a square with side lengths π centered at the origin. By a theorem known as the Schwarz Reflection Principle, we can fit pieces of Scherk's doubly periodic surface together horizontally and vertically to get a checkerboard domain (See Figure 2.21). Because one piece of this surface can be repeated or tiled in two directions, it is called a doubly periodic surface. It turns out that there are many examples of periodic minimal surfaces, and this is currently an active area of research.

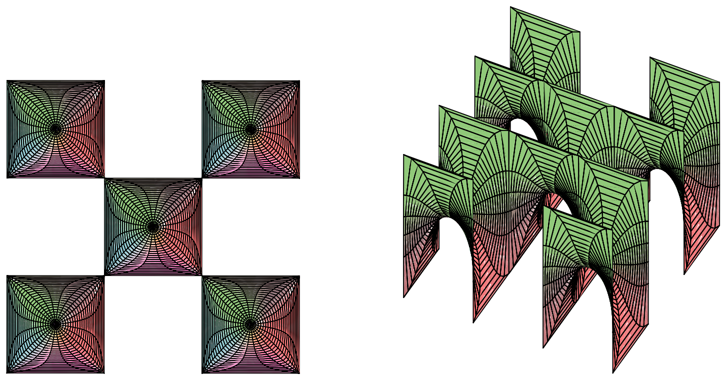


FIGURE 2.21. A tiling of Scherk's doubly periodic surface.

Let's summarize the minimal surfaces we have encountered so far, as well as a few other well-known examples.

(1) The plane:
$$\mathbf{x}(u,v) = (u, v, 0).$$
(2) The Enneper surface:
$$\mathbf{x}(u,v) = \left(u - \tfrac{1}{3}u^3 + uv^2, v - \tfrac{1}{3}v^3 + u^2v, u^2 - v^2\right).$$
(3) The catenoid:
$$\mathbf{x}(u,v) = (a\cosh v \cos u, a\cosh v \sin u, av).$$
(4) The helicoid:
$$\mathbf{x}(u,v) = (a\sinh v \cos u, a\sinh v \sin u, au).$$
(5) Scherk's doubly periodic surface:
$$\mathbf{x}(u,v) = \left(u, v, \ln\left(\frac{\cos u}{\cos v}\right)\right).$$
(6) Scherk's singly periodic surface:
$$\mathbf{x}(u,v) = (\operatorname{arcsinh}(u), \operatorname{arcsinh}(v), \arcsin(uv)).$$
(7) Henneberg surface:
$$\mathbf{x}(u,v) = (-1 + \cosh(2u)\cos(2v), -\sinh(u)\sin(v) - \tfrac{1}{3}\sinh(3u)\sin(3v),$$
$$-\sinh(u)\cos(v) + \tfrac{1}{3}\sinh(3u)\cos(3v)).$$
(8) Catalan surface:
$$\mathbf{x}(u,v) = (1 - \cos(u)\cosh(v), 4\sin(\tfrac{u}{2})\sinh(\tfrac{v}{2}), u - \sin(u)\cosh(v)).$$

In addition to the Enneper surface, the Henneberg surface and the Catalan surface are not embedded.

There is a fairly extensive list of minimal surfaces, and we have no way of listing all of them. So, instead, we often focus on trying to classify minimal surfaces. This means, we try to find results that include all possibilities for minimal surfaces with specific properties. The simplest example of this is Theorem 2.39.

THEOREM 2.39. Any nonplanar minimal surface in $\mathbb{R}^3$ that is also a surface of revolution is contained in a catenoid.

As we have seen, determining if a surface is minimal basically involves solving second order differential equations. We can simplify these equations if we use a specific type of parametrization of a surface known as an isothermal parametrization.

DEFINITION 2.40. A parametrization $\mathbf{x}$ is *isothermal* if $E = \mathbf{x}_u \cdot \mathbf{x}_u = \mathbf{x}_v \cdot \mathbf{x}_v = G$ and $F = \mathbf{x}_u \cdot \mathbf{x}_v = 0$.

What does this mean? Recall that E, F, and G describe how lengths on a surface are distorted as compared to their usual measurements in $\mathbb{R}^3$. So if $F = \mathbf{x}_u \cdot \mathbf{x}_v = 0$ then the vectors $\mathbf{x}_u$ and $\mathbf{x}_v$ are orthogonal and if $E = G$, then the amount of distortion is the same in these two orthogonal directions. Thus, we can think of an isothermal parametrization as mapping a small square in the domain to a small square on the surface. Sometimes an isothermal parametrization is called a conformal parametrization, because the angle between a pair of curves in the domain is equal to the angle between the corresponding pair of curves on the surface.

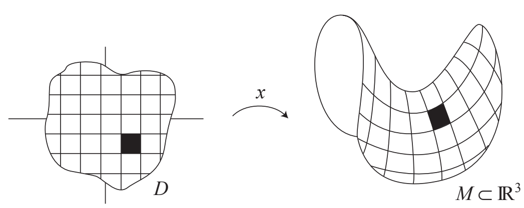


FIGURE 2.22. An isothermal parametrization maps small squares to small squares.

EXAMPLE 2.41. The parametrization

$$\mathbf{x}(u, v) = (a \cosh v \cos u, a \cosh v \sin u, av)$$

for the catenoid is isothermal, because in Example 2.35 we derived that $E = a^2 \cosh^2 v = G$ and $F = 0$. We can get a geometric sense that this parametrization is isothermal by using *DiffGeomTool* to graph this parametrization of the catenoid. Open *DiffGeomTool* and enter this parametrization with $a = 1$ and click the **Graph** button. Use the `Transparency of surface` and the `Transparency of the frame` sliders and click off and on the `Wireframe` button to make the curves more distinct. Notice that the grid of squares in the domain are pretty much mapped to a grid of squares as predicted above (see Figure 2.23).

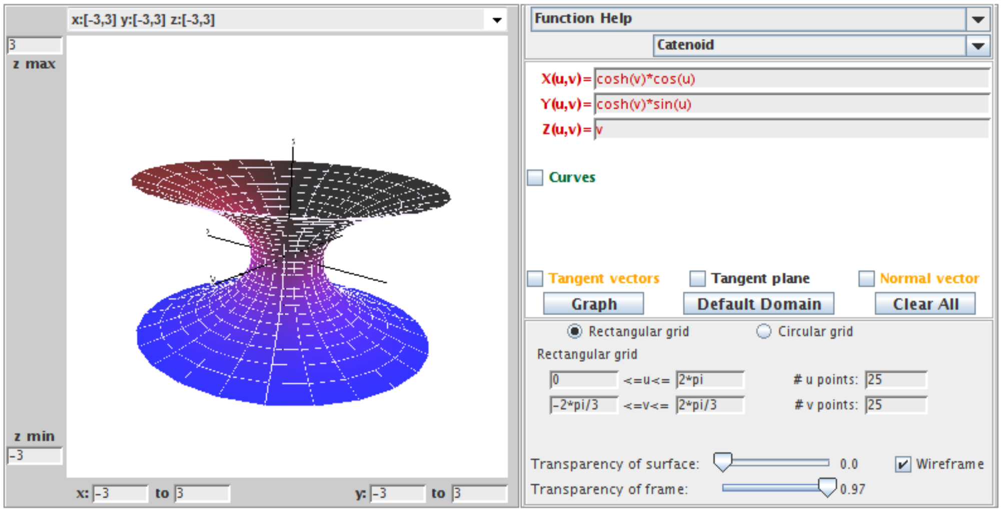


FIGURE 2.23. This parametrization of the catenoid is isothermal.

EXAMPLE 2.42. The parametrization

$$\mathbf{x}(u,v) = \big((a + b\cos v)\cos u, (a + b\cos v)\sin u, b\sin v\big)$$

for the torus is not isothermal. This is because in Exercise 2.32, you derived that

$$\begin{aligned} E =& (a + b\cos v)^2, \\ F =& 0, \\ G =& b^2. \end{aligned}$$

Because $F = 0$, the vectors $\mathbf{x}_u$ and $\mathbf{x}_v$ are orthogonal on the torus. But $E \geq G$ with equality only when $v = \pi + 2\pi k$, $(k \in \mathbb{Z})$ and either $a = 0$ or $a = 2b$. Thus the image of squares in the domain will be nonsquare rectangles whenever $v \neq \pi + 2\pi k$. Again, open *DiffGeomTool* and enter this parametrization for the torus with $a = 2$ and $b = 1$. Set the `Rectangular grid` values to 0 <= u <= 2∗ pi and 0 <= v <= 2∗ pi. Again, use the `Transparency of surface:` and the `Transparency of the frame:` sliders and click off and on the `Wireframe` button to make the curves more distinct. Notice that the grid of squares in the domain are mapped to a grid of mostly nonsquare rectangles as mentioned above. The ratio, $\frac{length}{height}$, of the sides of the rectangles is largest for the part of the torus farthest away from the origin. This occurs when $v = 0$ (or $v = 2\pi$) resulting in $E = 4$ while $G = 1$. On the other hand, the rectangles are squares for the part of the torus closest to the origin. This occurs when $v = \pi$ resulting in $E = 1$ while $G = 1$. This helps us see why this parametrization of the torus is not isothermal (see Figure 2.24).

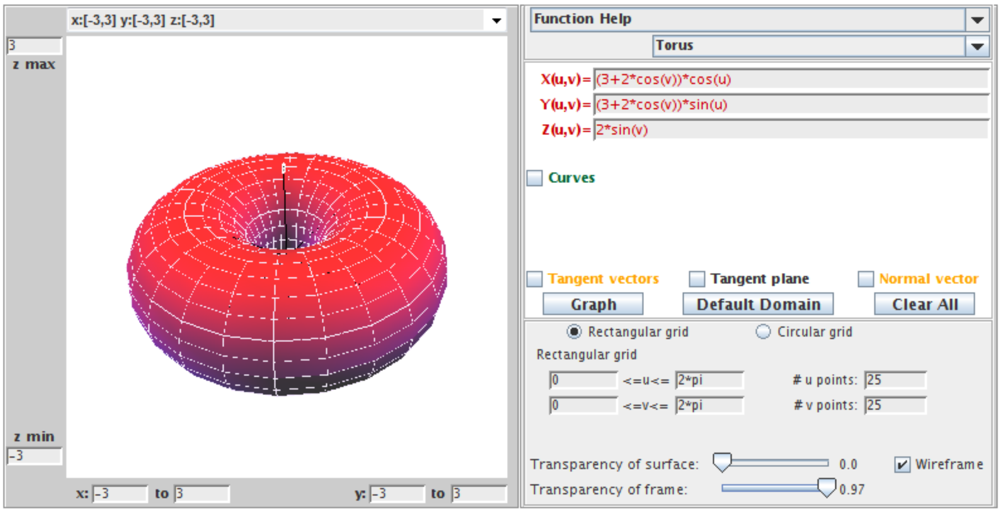


FIGURE 2.24. This parametrization of the torus is not isothermal.

EXERCISE 2.43. Using Definition 2.40 determine which of the following parametrizations of minimal surfaces is isothermal:

(a) The Enneper surface parametrized by
$$\mathbf{x}(u,v) = \left(u - \tfrac{1}{3}u^3 + uv^2, v - \tfrac{1}{3}v^3 + u^2 v, u^2 - v^2\right);$$
(b) Scherk's doubly periodic surface parametrized by
$$\mathbf{x}(u,v) = \left(u, v, \ln\left(\frac{\cos u}{\cos v}\right)\right);$$
(c) The helicoid parametized by
$$\mathbf{x}(u,v) = (a \sinh v \cos u, a \sinh v \sin u, au).$$

Try it out!

EXPLORATION 2.44. Use *DiffGeomTool* to check the reasonableness of your answers in Exercise 2.43 by graphing each parametrization in that exercise as was done in Examples 2.41 and 2.42. Set the # u points: to 20 and the # v points: to 20, and use the following values for the `U/V domain` boxes:

(a) Enneper surface:
$-\frac{\pi}{3} <=$ u,v $<= \frac{\pi}{3}$;
(b) Scherk's doubly periodic surface:
$-\frac{\pi}{2} + 0.1 <=$ u,v $<= \frac{\pi}{2} - 0.1$;
(c) Helicoid:
$-\pi <=$ u,v $<= \pi$.

Try it out!

From Exercise 2.43 and Exploration 2.44 you have seen there are parametrizations of minimal surfaces that are not isothermal. However, requiring minimal surfaces to have an isothermal parametrization is not a restriction because of the following theorem.

THEOREM 2.45. Every minimal surface in $\mathbb{R}^3$ has an isothermal parametrization.

REMARK 2.46. See [**23**] for a proof of Theorem 2.45. In fact, every differentiable surface has an isothermal parametrization. This is a very interesting result. Unfortunately, a proof of this is beyond the scope of this text, but if you are interested, a proof is given in [**2**], pp 15-35.

Recall that in Example 2.35, we derived that the isothermal parametrization for the catenoid

$$\mathbf{x}(u,v) = \Big(a\cosh v\cos u, a\cosh v\sin u, av\Big)$$

has

$$e = -g.$$

In general, we have the following result.

THEOREM 2.47. Let M be a surface with isothermal parametrization. Then M is minimal if and only if $e = -g$.

EXERCISE 2.48. Prove Theorem 2.47.
Try it out!

EXPLORATION 2.49. Recall that $e = -g$ for the coefficients of the 2nd fundamental form represents that the u-parameter curve and the v-parameter curve are bending the same amount away from the normal $\mathbf{n}$ but in different directions. Use *DiffGeomTool* in connection with Theorem 2.47 to geometrically verify which of the following surfaces are minimal:

(a) Enneper surface:
$$\mathbf{x}(u,v) = \Big(u - \tfrac{1}{3}u^3 + uv^2, v - \tfrac{1}{3}v^3 + u^2v, u^2 - v^2\Big);$$
(b) Cylinder:
$$\mathbf{x}(u,v) = \big(\cos u, \sin u, v\big);$$
(c) Helicoid:
$$\mathbf{x}(u,v) = (a\sinh v\cos u, a\sinh v\sin u, av).$$

Try it out!

Now, here is an interesting and important result that brings in an idea from complex analysis. Recall from complex analysis that if $f(z) = x(u,v) + iy(u,v)$ is an analytic function, then the Cauchy-Riemann equations hold for f. That is,

$$x_u = y_v, \qquad x_v = -y_u.$$

In such a case, y is called the *harmonic conjugate* of x. Also, if f is analytic, then

$$f'(z) = x_u + iy_u. \tag{8}$$

This concept allows us to relate a minimal surface to another minimal surface, known as its conjugate minimal surface.

Definition 2.50. Let $\mathbf{x}$ and $\mathbf{y}$ be isothermal parametrizations of minimal surfaces such that their component functions are pairwise harmonic conjugates. That is,

$$\mathbf{x}_u = \mathbf{y}_v \qquad \text{and} \qquad \mathbf{x}_v = -\mathbf{y}_u. \tag{9}$$

In such a case, $\mathbf{x}$ and $\mathbf{y}$ are called *conjugate minimal surfaces.*

Example 2.51. Let's find the conjugate surface of the catenoid parametrized by

$$\mathbf{x}(u,v) = (a\cosh v\cos u, a\cosh v\sin u, av).$$

Let $\mathbf{y}(u,v)$ be the parametrization of this conjugate surface. By the first part of eq (9), we know

$$\mathbf{y}_v = \mathbf{x}_u = (-a\cosh v\sin u, a\cosh v\cos u, 0).$$

Integrating this with respect to v yields

$$\mathbf{y} = (-a\sinh v\sin u + F_1(u), a\sinh v\cos u + F_2(u), F_3(u)),$$

where each $F_k(u)$ is a function independent of v. Similarly, by the second part of eq (9), we derive

$$\mathbf{y} = (-a\sinh v\sin u + G_1(v), a\sinh v\cos u + G_2(v), -au + G_3(v)).$$

Equating these two expressions for $\mathbf{y}$ we get that

$$\mathbf{y} = (-a\sinh v\sin u + K_1, a\sinh v\cos u + K_2, -au + K_3).$$

Using the substitution $u = \widetilde{u} - \frac{\pi}{2}$, $v = \widetilde{v}$, and letting $K_1 = 0$, $K_2 = 0$, and $K_3 = -a\frac{\pi}{2}$, does not affect the geometry of this minimal surface, and yields the parametrization of a helicoid

$$\mathbf{y}(\widetilde{u},\widetilde{v}) = (a\sinh\widetilde{v}\cos\widetilde{u}, a\sinh\widetilde{v}\sin\widetilde{u}, -a\widetilde{u})$$

given in Exercise 2.24 (Note that the negative sign in the third component function just has the effect of reflecting the surface through the x_1x_2-plane). Hence, the conjugate surface of this catenoid is a helicoid.

This idea of conjugate minimal surfaces gets really interesting. It turns out that any two conjugate minimal surfaces can be joined through a one-parameter family of minimal surfaces by the equation

$$\mathbf{z} = (\cos t)\mathbf{x} + (\sin t)\mathbf{y},$$

where $t \in \mathbb{R}$. Note that when $t = 0$ we have the minimal surface parametrized by $\mathbf{x}$, and when $t = \frac{\pi}{2}$ we have the minimal surface parametrized by $\mathbf{y}$. So for $0 \le t \le \frac{\pi}{2}$, we have a continuous parameter of minimal surfaces known as *associated surfaces.* In other words, we can continuously "morph" one minimal surface into another minimal surface so that all the in-between surfaces are also minimal.

In Example 2.51, we saw that the helicoid and the catenoid are conjugate surfaces. Images of them and certain associated surfaces are shown in Figure 2.25.

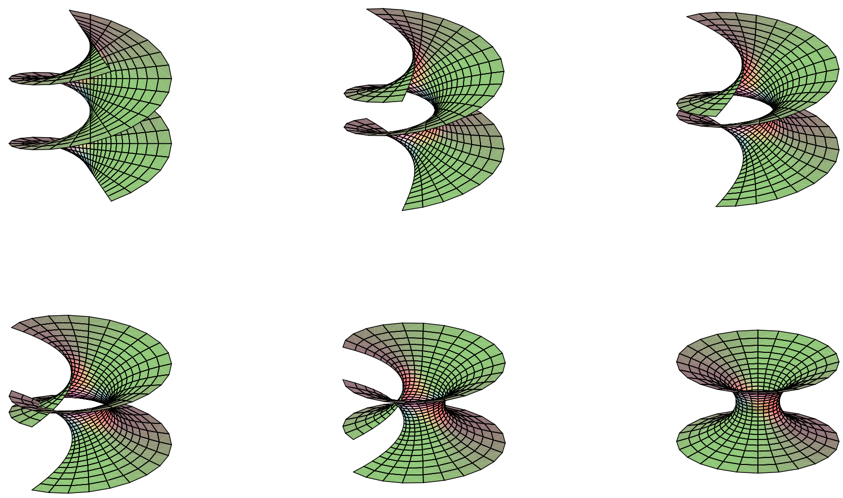


FIGURE 2.25. The helicoid, some associated surfaces, and the catenoid.

This is neat, but it is just the beginning. The rest of this section will explore properties of conjugate surfaces.

EXERCISE 2.52. Find the conjugate minimal surface for the Enneper surface

$$\mathbf{x}(u,v) = \left(u - \frac{1}{3}u^3 + uv^2, v - \frac{1}{3}v^3 + u^2v, u^2 - v^2\right).$$

Try it out!

If we try to determine the conjugate minimal surface for Scherk's doubly periodic surface with the parametrization

$$\mathbf{x}(u,v) = \left(u, v, \ln\left(\frac{\cos u}{\cos v}\right)\right),$$

this method will not work, because this parametrization is not isothermal. However, later we will see that Scherk's doubly periodic surface does have a conjugate surface. It is Scherk's singly periodic surface (see Figure 2.26).

EXPLORATION 2.53. You can see the associated surfaces that occur between Scherk's doubly periodic surface and Scherk's singly periodic surface by using another applet for this chapter. This applet is called *MinSurfTool*, and it can be used to visualize and explore minimal surfaces in $\mathbb{R}^3$. Open the *MinSurfTool* applet. On the right-hand side near the top, there is a set of tabs for different features of this applet. For this exploration, we want to use the **W.E.(p,q)** feature, so make sure that the **W.E.(p,q)** tab is on top (if it is on top, it should be a different color than the other tabs). In the

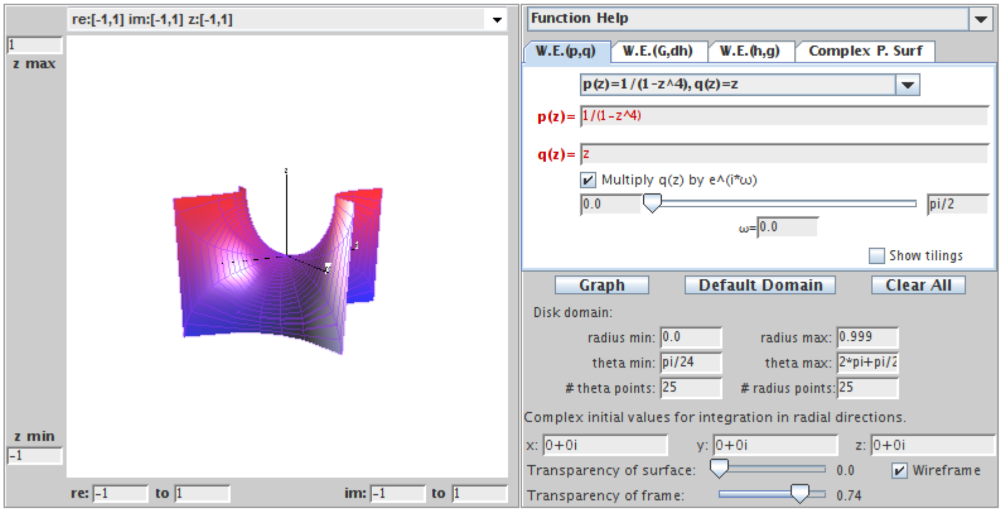


FIGURE 2.26. Scherk's singly periodic surface.

Pre-set functions window, choose $p(z) = 1/(1-z^4)$, $q(z) = z$. Click on the **Graph** button, and one piece of Scherk's singly periodic surface will appear. Then move the slider arrow, that is above the **Graph** button, to the right to see how Scherk's singly periodic is transformed by way of the associated surfaces into Scherk's doubly periodic surface.

Try it out!

Recall that individual pieces of Scherk's doubly periodic surface can be put together in the x_1x_2-plane in a checkerboard fashion. So, these pieces repeat (or are periodic) in two directions, x_1 and x_2. Surfaces that repeat in one direction are called singly periodic. For example, individual pieces of Scherk's singly periodic surface can fit together creating a tower in the x_3 direction. You can visualize adding two pieces together by taking one piece of Scherk's singly periodic surface and adding it to another piece that has been reflected across the x_1x_2-plane and shifted up in the x_3 direction. By continuing to do this, you can create a tower of several pieces (see Figure 2.27). Note that the helicoid is a singly periodic surface too.

Earlier in Example 2.31 we saw that the coefficients of the first fundamental form for the given parametrization of a catenoid are $E = a^2\cosh^2 v$, $F = 0$, and $G = a^2\cosh^2 v$. In Exercise 2.140 from the **Additional Exercises** at the end of the chapter it can be shown that for the given parametrization of Enneper's surface these coefficients are $E = (1+u^2+v^2)^2$, $F = 0$, and $G = (1+u^2+v^2)^2$. Clearly, the E's and G's do not match up. However, for any two conjugate minimal surfaces and their associated minimal surfaces, the coefficients of the first fundamental form are always the same. The following exercise will help you prove this surprising result.

EXERCISE 2.54.

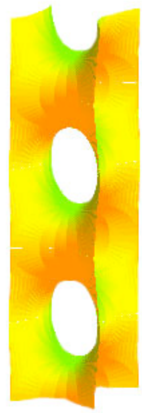


FIGURE 2.27. Scherk's singly periodic surface.

(a) Prove that given two conjugate minimal surfaces, $\mathbf{x}$ and $\mathbf{y}$, all surfaces of the one-parameter family

$$\mathbf{z} = (\cos t)\mathbf{x} + (\sin t)\mathbf{y}$$

have the same fundamental form: $E = \mathbf{x}_u \cdot \mathbf{x}_u = \mathbf{y}_u \cdot \mathbf{y}_u$, $F = 0$, $G = \mathbf{x}_v \cdot \mathbf{x}_v = \mathbf{y}_v \cdot \mathbf{y}_v$.

(b) Prove that all the surfaces in the one-parameter family $\mathbf{z}$ from part (a) are minimal for all $t \in \mathbb{R}$.

Try it out!

Finally, recall that the normal vector $\mathbf{n}$ at a point on a surface points orthogonally away from the surface. Since different minimal surfaces have different shapes, there is no reason to suspect that the normal vectors on one surface will be related to the normal vectors on another surface. However, for conjugate minimal surfaces and their associated minimal surfaces there is a strong connection. It turns out that for any point in the domain, the corresponding surface normal points in the same direction on all these minimal surfaces. The next theorem establishes this idea.

THEOREM 2.55. Let $\mathbf{x}, \mathbf{y} : D \to \mathbb{R}^3$ be isothermal parametrizations of conjugate minimal surfaces. Then for each $(u_0, v_0) \in D$, the corresponding surface unit normal is the same for all the associated surfaces.

PROOF. Let $(u_0, v_0) \in D$. Let $\mathbf{n}^{\mathbf{x}}$ and $\mathbf{n}^{\mathbf{y}}$ represent the surface normal for $\mathbf{x}$ and for $\mathbf{y}$, respectively. Then by the definition of conjugate surfaces, $\mathbf{x}$ and $\mathbf{y}$ have the same unit normal, because

$$\mathbf{n}^{\mathbf{x}} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|} = \frac{\mathbf{y}_v \times -\mathbf{y}_u}{|\mathbf{y}_v \times -\mathbf{y}_u|} = \frac{\mathbf{y}_u \times \mathbf{y}_v}{|\mathbf{y}_u \times \mathbf{y}_v|} = \mathbf{n}^{\mathbf{y}}.$$

To show that this is true for the associated surfaces , let $\mathbf{z} = (\cos t)\mathbf{x} + (\sin t)\mathbf{y}$ be the parametrization of the associated surfaces. Then

$$\begin{aligned}
\mathbf{z}_u \times \mathbf{z}_v =& \big(\cos t\mathbf{x}_u + \sin t\mathbf{y}_u\big) \times \big(\cos t\mathbf{x}_v + \sin t\mathbf{y}_v\big)\\
=& \cos^2 t\big(\mathbf{x}_u \times \mathbf{x}_v\big) + \cos t \sin t\big(\mathbf{x}_u \times \mathbf{y}_v\big) + \cos t \sin t\big(\mathbf{y}_u \times \mathbf{x}_v\big) + \sin^2 t\big(\mathbf{y}_u \times \mathbf{y}_v\big)\\
=& \cos^2 t\big(\mathbf{x}_u \times \mathbf{x}_v\big) + \cos t \sin t\big(\mathbf{x}_u \times \mathbf{x}_u\big) + \cos t \sin t\big(-\mathbf{x}_v \times \mathbf{x}_v\big) + \sin^2 t\big(-\mathbf{x}_v \times \mathbf{x}_u\big)\\
=& \cos^2 t\big(\mathbf{x}_u \times \mathbf{x}_v\big) + \cos t \sin t\big(\mathbf{0}\big) + \cos t \sin t\big(\mathbf{0}\big) + \sin^2 t\big(\mathbf{x}_u \times \mathbf{x}_v\big)\\
=& \mathbf{x}_u \times \mathbf{x}_v.
\end{aligned}$$

$\square$

The following example and exploration helps us visualize this idea.

EXAMPLE 2.56. Using *DiffGeomTool* we can graph the catenoid and its conjugate surface, the helicoid, whose parametrizations are given in Example 2.51. If we plot the normal $\mathbf{n}$ at the point $(\frac{\pi}{3}, -\frac{\pi}{4})$ on these conjugate surfaces, we see that both normals point in the same direction as guaranteed by Theorem 2.55 (see Figure 2.28 and Figure 2.29).

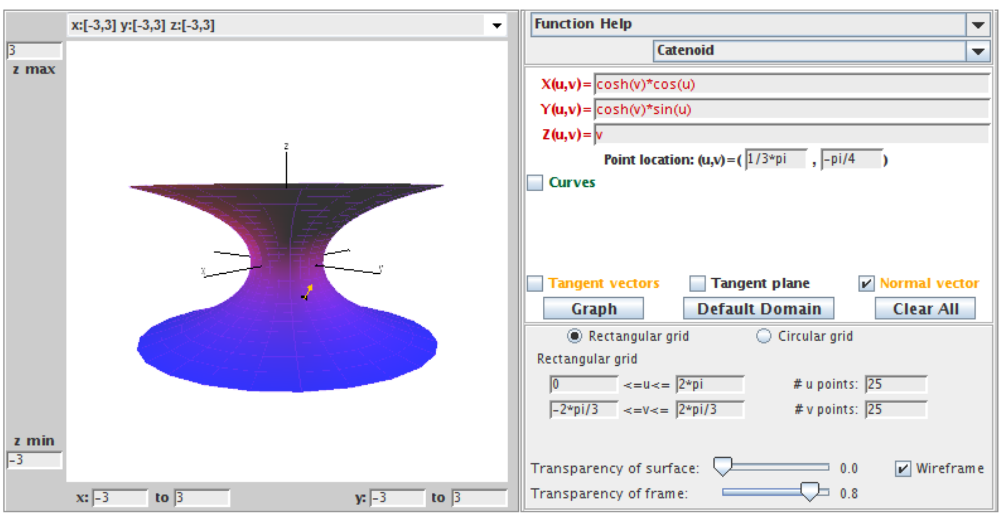


FIGURE 2.28. The catenoid with $\mathbf{n}$ at $(\frac{\pi}{3}, -\frac{\pi}{4})$.

EXPLORATION 2.57. Open two separate windows of *DiffGeomTool*. In one plot the catenoid parametrized by

$$\mathbf{x}(u, v) = (\cosh v \cos u, \cosh v \sin u, v),$$

where $0 \leq u \leq 2\pi$ and $-\frac{2\pi}{3} \leq v \leq \frac{2\pi}{3}$. In the other plot its conjugate surface, the helicoid, with parametrization

$$\mathbf{y}(u, v) = (-\sinh v \sin u, \sinh v \cos u, -u),$$

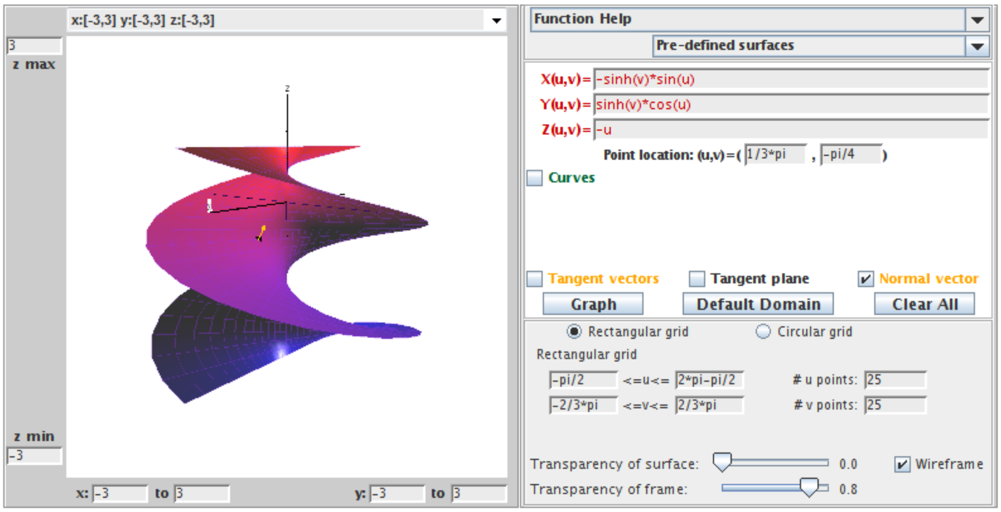


FIGURE 2.29. The conjugate helcoid with $\mathbf{n}$ at $(\frac{\pi}{3}, -\frac{\pi}{4})$.

where $-\frac{\pi}{2} \leq u \leq 2\pi - \frac{\pi}{2}$ and $-\frac{2\pi}{3} \leq v \leq \frac{2\pi}{3}$ (Note that the u values for the helicoid are different than the values for the catenoid because we used the substitution $u = \widetilde{u} - \frac{\pi}{2}$ in Example 2.51). Make sure that the x, y, z-axes are positioned in the same directions. Then plot the following unit normals, $\mathbf{n}$, on each surface at the following points and observe that $\mathbf{n}$ points in the same direction as prescribed by Theorem 2.55:

(a) at $(\frac{\pi}{3}, 0)$, (b) at $(\frac{\pi}{4}, -\frac{\pi}{2})$, (c) at $(\frac{\pi}{6}, \frac{\pi}{2})$.

Try it out!

2.4. Weierstrass Representation

At the end of the last section, we saw that we could apply complex analysis to minimal surface theory to define conjugate surfaces. In this chapter we will again use complex analysis to learn more about minimal surfaces. First, we will use inherent properties of an isothermal parametrization to give us a necessary and sufficient condition for a surface to be minimal. This condition is very useful for finding and classifying minimal surfaces, as we will see. First, we state a theorem which will lead to the desired condition.

THEOREM 2.58. If the parametrization $\mathbf{x}$ is isothermal, then

$$\mathbf{x}_{uu} + \mathbf{x}_{vv} = 2EH\mathbf{n},$$

where E is a coefficient of the first fundamental form and H is the mean curvature.

EXERCISE 2.59 (Proof of Theorem 2.58). The set $\{\mathbf{x}_u, \mathbf{x}_v, \mathbf{n}\}$ forms a basis for $\mathbb{R}^3$. Assume $F = 0$. Then the vector $\mathbf{x}_{uu}$ can be expressed in terms of these bases vectors.

That is,

$$\mathbf{x}_{uu} = \Gamma^u_{uu}\mathbf{x}_u + \Gamma^v_{uu}\mathbf{x}_v + e\mathbf{n},$$

where the coefficients, Γ^u_{uu} and Γ^v_{uu}, are known as Christoffel symbols and e comes from the coefficient of the second fundamental form. That is, $e = \mathbf{n} \cdot \mathbf{x}_{uu}$.

(a) Show that $\Gamma^u_{uu} = \frac{E_u}{2E}$ and $\Gamma^v_{uu} = -\frac{E_v}{2G}$ by taking the inner product of $\mathbf{x}_{uu}$ with each of the basis vectors. In a similar manner, it can be shown that

$$\mathbf{x}_{vv} = -\frac{G_u}{2E}\,\mathbf{x}_u + \frac{G_v}{2G}\,\mathbf{x}_v + g\mathbf{n}.$$

(b) Use the mean curvature equation (5) and the results from (a) to show that if the parametrization $\mathbf{x}$ is isothermal, then

$$\mathbf{x}_{uu} + \mathbf{x}_{vv} = 2EH\mathbf{n}.$$

Try it out!

Now, where do we go from here? Recall that for a minimal surface, $H \equiv 0$. So Theorem 2.58 tells us that $\mathbf{x}_{uu} + \mathbf{x}_{vv} \equiv 0$. But what does this last equation represent? It is Laplace's equation and relates to harmonic functions. Recall that $\varphi(u, v)$ is a real-valued *harmonic function* if $\varphi_{uu} + \varphi_{vv} = 0$ (for example, $\varphi(u, v) = u^2 - v^2$ is harmonic). This leads to the condition for a surface to be minimal that we referred to in the beginning of this section. The corollary below provides the final piece of the puzzle that will allow us to explicitly describe minimal surfaces using the Weierstrass representation, which we will derive in this section.

Corollary 2.60. A surface M with an isothermal parametrization $\mathbf{x}(u, v) = \big(x_1(u, v), x_2(u, v), x_3(u, v)\big)$ is minimal if and only if x_1, x_2, x_3 are harmonic.

Make sure you understand the significance of this result. First, we need an isothermal parametrization for our surface, but this is not a difficulty because of Theorem 2.45. Then this result tells us we will have a minimal surface if and only if the coordinate functions of that parametrization are harmonic functions. This will provide us another way to create and to prove a surface is minimal.

Proof. ($\Rightarrow$) If M is minimal, then $H = 0$ and so by Theorem 2.58 $\mathbf{x}_{uu} + \mathbf{x}_{vv} = 0$, and hence the coordinate functions are harmonic. ($\Leftarrow$) Suppose x_1, x_2, x_3 are harmonic. Then $\mathbf{x}_{uu} + \mathbf{x}_{vv} = 0$. So by Theorem 2.58 we have that $2(\mathbf{x}_u \cdot \mathbf{x}_u)H\mathbf{n} = 0$. But $\mathbf{n} \neq 0$ and $E = \mathbf{x}_u \cdot \mathbf{x}_u \neq 0$. Hence, $H = 0$ and M is minimal. □

Exercise 2.61. Given the parametrization for the Enneper surface

$$\mathbf{x}(u, v) = \left(u - \frac{1}{3}u^3 + uv^2, v - \frac{1}{3}v^3 + u^2 v, u^2 - v^2\right),$$

use Corollary 2.60 to prove that the Enneper surface is a minimal surface.

Try it out!

The importance of Corollary 2.60 is not in proving specific surfaces are minimal. Instead, it lies in establishing a general formula that will guarantee any surface created by it will be minimal. This formula is known as the Weierstrass representation for minimal surfaces. This is neat, because it will provide us with a simple way to construct a lot of examples of minimal surfaces using functions from complex analysis. After stating the Weierstrass representation in Theorem 2.66 we will use the rest of this section to create more minimal surfaces. However, it turns out that not all minimal surfaces are of equal interest. So in section 2.5 of this chapter we leave behind the idea of creating arbitrary minimal surfaces and instead explore properties that make certain minimal surfaces more interesting.

Now we will derive this important formula, the Weierstrass Representation, and bring in the connection with complex analysis. Suppose M is a minimal surface with an isothermal parametrization $\mathbf{x}(u,v)$. Let $z = u+iv$ be a point in the complex plane. Recall that $\overline{z} = u - iv$ is the conjugate of z. Using these representations for z and $\overline{z}$ we can solve for u, v in terms of $z, \overline{z}$ to get

$$u = \frac{z+\overline{z}}{2} \text{ and } v = \frac{z-\overline{z}}{2i}.$$

Then the parametrization of the minimal surface M can be written in terms of the complex variables z and $\overline{z}$ as:

$$\mathbf{x}(z,\overline{z}) = \big(x_1(z,\overline{z}), x_2(z,\overline{z}), x_3(z,\overline{z})\big).$$

Exercise 2.62. Let $f(u,v) = x(u,v) + iy(u,v)$ be a complex function. Using the notation $u = \frac{z+\overline{z}}{2}$ and $v = \frac{z-\overline{z}}{2i}$, we can express f in terms of z and $\overline{z}$ instead of u and v. That is, we have the function $f(z,\overline{z})$. In this exercise you will prove the neat result that f is analytic if and only if f can be written in terms of $z = u+iv$ alone without using $\overline{z} = u - iv$.

(a) Using the chain rule, derive the following formulas:

$$\begin{aligned}\frac{\partial f}{\partial z} =& \frac{1}{2}\left(\frac{\partial x}{\partial u} + \frac{\partial y}{\partial v}\right) + \frac{i}{2}\left(\frac{\partial y}{\partial u} - \frac{\partial x}{\partial v}\right),\\ \frac{\partial f}{\partial \overline{z}} =& \frac{1}{2}\left(\frac{\partial x}{\partial u} - \frac{\partial y}{\partial v}\right) + \frac{i}{2}\left(\frac{\partial y}{\partial u} + \frac{\partial x}{\partial v}\right).\end{aligned}$$

(b) Show that f is analytic $\Longleftrightarrow \frac{\partial f}{\partial \overline{z}} = 0$.

Try it out!

Example 2.63. The function $f_1(z) = z^2$ is analytic, because $\frac{\partial f_1}{\partial \overline{z}}(z^2) = 0$. However, $f_2(z) = |z|^2 = z\overline{z}$ is not analytic, because $\frac{\partial f_1}{\partial \overline{z}} = z \neq 0$.

Exercise 2.64. Prove that

$$4\left(\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \overline{z}}\right)\right) = f_{uu} + f_{vv}. \tag{10}$$

Try it out!

The next theorem expands upon Corollary 2.60 to establish the Weierstrass representation for minimal surfaces.

THEOREM 2.65. Let M be a surface with parametrization $\mathbf{x} = (x_1, x_2, x_3)$ and let $\phi = (\varphi_1, \varphi_2, \varphi_3)$, where $\varphi_k = \frac{\partial x_k}{\partial z}$. Then $\mathbf{x}$ is isothermal $\Longleftrightarrow \phi^2 = (\varphi_1)^2+(\varphi_2)^2+(\varphi_3)^2 = 0$. If $\mathbf{x}$ is isothermal, then M is minimal $\Longleftrightarrow$ each φ_k is analytic.

Before we prove Theorem 2.65, let's look at applying it to a specific example to help us better understand what the theorem is saying. Suppose we have the parametrization $\mathbf{x} = (x_1, x_2, x_3) = (z-\frac{1}{3}z^3, -i(z+\frac{1}{3}z^3), z^2)$. Then $\varphi_1 = \frac{\partial x_1}{\partial z} = 1-z^2$, $\varphi_2 = \frac{\partial x_2}{\partial z} = -i(1+z^2)$, and $\varphi_3 = \frac{\partial x_3}{\partial z} = 2z$. Notice that $\phi^2 = [1-z^2]^2+[-i(1+z^2)]^2+[2z]^2 = 0$. Thus, by the theorem, the parametrization $\mathbf{x}$ is isothermal. Also, each φ_k is a polynomial and hence analytic. So $\mathbf{x}$ is a parametrization of a minimal surface (in fact, it is Enneper's surface). Make sure you understand how this example relates to Theorem 2.65 before you read the following proof of the theorem.

PROOF. Applying the complex differential operator $\frac{\partial f}{\partial z}$ from Exercise 2.62 to this situation and then squaring the terms, we have $(\varphi_k)^2 = \left(\frac{\partial x_k}{\partial z}\right)^2 = \left[\frac{1}{2}(\frac{\partial x_k}{\partial u} - i\frac{\partial x_k}{\partial v})\right]^2 = \frac{1}{4}\left[(\frac{\partial x_k}{\partial u})^2 - (\frac{\partial x_k}{\partial v})^2 - 2i\frac{\partial x_k}{\partial u}\frac{\partial x_k}{\partial v}\right]$. Also, recall that $\mathbf{x_u}\cdot\mathbf{x_u} = (\frac{\partial x_1}{\partial u})^2 + (\frac{\partial x_2}{\partial u})^2 + (\frac{\partial x_3}{\partial u})^2 = \sum_{k=1}^{3}(\frac{\partial x_k}{\partial u})^2$ and similarly $\mathbf{x_v}\cdot\mathbf{x_v} = \sum_{k=1}^{3}(\frac{\partial x_k}{\partial v})^2$. Hence,

$$\begin{aligned}\phi^2 =&(\varphi_1)^2 + (\varphi_2)^2 + (\varphi_3)^2\\ =&\frac{1}{4}\left[\sum_{k=1}^{3}\left(\frac{\partial x_k}{\partial u}\right)^2 - \sum_{k=1}^{3}\left(\frac{\partial x_k}{\partial v}\right)^2 - 2i\sum_{k=1}^{3}\frac{\partial x_k}{\partial u}\frac{\partial x_k}{\partial v}\right]\\ =&\frac{1}{4}\big(\mathbf{x}_u\cdot\mathbf{x}_u - \mathbf{x}_v\cdot\mathbf{x}_v - 2i(\mathbf{x}_u\cdot\mathbf{x}_v)\big)\\ =&\frac{1}{4}(E-G-2iF).\end{aligned}$$

Thus, $\mathbf{x}$ is isothermal $\Longleftrightarrow E = G, F = 0 \Longleftrightarrow \phi^2 = 0$.

Now suppose that $\mathbf{x}$ is isothermal. By Corollary 2.60, it suffices to show that for each k, x_k is harmonic $\Longleftrightarrow \varphi_k$ is analytic. Using eq (10) and Exercise 2.62 this follows because

$$\frac{\partial^2 x_k}{\partial u\partial u} + \frac{\partial^2 x_k}{\partial v\partial v} = 4\left(\frac{\partial}{\partial \overline{z}}\left(\frac{\partial x_k}{\partial z}\right)\right) = 4\left(\frac{\partial}{\partial \overline{z}}\Big(\varphi_k\Big)\right) = 0.$$

□

Note that if $\mathbf{x}$ is isothermal

$$|\phi|^2 = \left|\frac{\partial x_1}{\partial z}\right|^2 + \left|\frac{\partial x_2}{\partial z}\right|^2 + \left|\frac{\partial x_3}{\partial z}\right|^2 = \frac{1}{4}\left(\sum_{k=1}^{3}\left(\frac{\partial x_k}{\partial u}\right)^2 + \sum_{k=1}^{3}\left(\frac{\partial x_k}{\partial v}\right)^2\right)$$
$$= \frac{1}{4}\left(\mathbf{x}_u \cdot \mathbf{x}_u + \mathbf{x}_v \cdot \mathbf{x}_v\right) = \frac{1}{4}(E+G) = \frac{E}{2}.$$

So if $|\phi|^2 = 0$, then all the coefficients of the first fundamental form are zero and M degenerates to a point. Similarly, we want $|\phi|^2$ to be finite.

Finally, we need to solve $\varphi_k = \frac{\partial x_k}{\partial z}$ for x_k since the parametrization of the surface is given as $\mathbf{x} = (x_1, x_2, x_3)$. The difficulty is that x_k is a function of two variables, z and $\overline{z}$, and we want to have a representation in which we only have to integrate with respect to one variable. To overcome this difficulty, we will use some ideas about differentials (see [**26**] for a nice introduction to differentials). First, since x_k is also a function of the two variables u and v, we can write

$$dx_k = \frac{\partial x_k}{\partial u}du + \frac{\partial x_k}{\partial v}dv. \tag{11}$$

Also, $dz = du + idv$. Using Exercise 2.62 we have

$$\varphi_k dz = \frac{\partial x_k}{\partial z}dz = \frac{1}{2}\left(\frac{\partial x_k}{\partial u} - i\frac{\partial x_k}{\partial v}\right)(du + idv)$$
$$= \frac{1}{2}\left[\frac{\partial x_k}{\partial u}du + \frac{\partial x_k}{\partial v}dv + i\left(\frac{\partial x_k}{\partial u}dv - \frac{\partial x_k}{\partial v}du\right)\right],$$
$$\overline{\varphi_k dz} = \overline{\varphi_k}\,\overline{dz} = \overline{\frac{\partial x_k}{\partial z}}\,\overline{dz} = \frac{1}{2}\left(\frac{\partial x_k}{\partial u} + i\frac{\partial x_k}{\partial v}\right)(du - idv)$$
$$= \frac{1}{2}\left[\frac{\partial x_k}{\partial u}du + \frac{\partial x_k}{\partial v}dv - i\left(\frac{\partial x_k}{\partial u}dv - \frac{\partial x_k}{\partial v}du\right)\right].$$

Adding these two equations yields

$$\frac{\partial x_k}{\partial u}du + \frac{\partial x_k}{\partial v}dv = \varphi_k dz + \overline{\varphi_k dz} = 2\,\mathrm{Re}\{\varphi_k dz\}. \tag{12}$$

Combining eq (11) and eq (12), we have

$$dx_k = 2\,\mathrm{Re}\{\varphi_k dz\}.$$

Therefore, $x_k = 2\,\mathrm{Re}\int \varphi_k dz + c_k$. Since adding c_k just translates the image by a constant amount and multiplying each coordinate function by 2 just scales the the surface, these constants do not affect the geometric shape of the surface. Hence, we do not need these constants and we will let our coordinate function be

$$x_k = \mathrm{Re}\int \varphi_k dz.$$

Summary: If we have analytic functions φ_k $(k = 1, 2, 3)$ such that

$$\phi^2 = 0 \qquad \text{and} \qquad |\phi|^2 \neq 0 \text{ and is finite,}$$

then the parametrization

$$\mathbf{x} = \Big(\operatorname{Re}\int \varphi_1(z)dz, \operatorname{Re}\int \varphi_2(z)dz, \operatorname{Re}\int \varphi_3(z)dz\Big) \tag{13}$$

defines a minimal surface.

For example, consider

$$\begin{aligned}
\varphi_1 =& p(1+q^2)\\
\varphi_2 =& -ip(1-q^2)\\
\varphi_3 =& -2ipq.
\end{aligned}$$

Then

$$\begin{aligned}
\phi^2 =&[p(1+q^2)]^2 + [-ip(1-q^2)]^2 + [-2ipq]^2\\
=&[p^2 + 2p^2q^2 + p^2q^4] - [p^2 - 2p^2q^2 + p^2q^4] - [4p^2q^2]\\
=&0,
\end{aligned}$$

and

$$\begin{aligned}
|\phi|^2 =&|p(1+q^2)|^2 + |-ip(1-q^2)|^2 + |-2ipq|^2\\
=&|p|^2[(1+q^2)(1+\ \overline{q}^2) + (1-q^2)(1-\ \overline{q}^2) + 4q\ \overline{q}]\\
=&|p|^2[2(1+2q\ \overline{q} + q^2\ \overline{q}^2)\\
=&4|p|^2(1+|q|^2)|^2 \neq 0 \qquad \text{(note: if } p = 0\text{, then } \varphi_k = 0 \text{ for all } k).
\end{aligned}$$

Notice that p, pq^2, and pq have to be analytic in order for each φ_k to be analytic. If p is analytic with a zero of order $2m$ at z_0, then q can have a pole of order no larger than m at z_0. Recall that a function that is analytic in a domain D except possibly at poles is known as a *meromorphic* function in D. This leads to the following result.

Theorem 2.66 (Weierstrass Representation (p,q)). Every regular minimal surface has a local isothermal parametric representation of the form

$$\begin{aligned}
\mathbf{x} =& (x_1(z), x_2(z), x_3(z))\\
=& \Big(\operatorname{Re}\Big\{\int_a^z p(1+q^2)dz\Big\},\\
&\qquad \operatorname{Re}\Big\{\int_a^z -ip(1-q^2)dz\Big\},\\
&\qquad\qquad \operatorname{Re}\Big\{\int_a^z -2ipqdz\Big\}\Big),
\end{aligned}$$

where p is an analytic function and q is a meromorphic function in some domain $\Omega \subset \mathbb{C}$, having the property that at each point where q has a pole of order m, p has a zero of order at least $2m$, and $a \in \Omega$ is a constant.

EXAMPLE 2.67. For $p(z) = 1, q(z) = iz$, we get

$$\begin{aligned}\mathbf{x} &= \left(\mathrm{Re}\left\{\int_0^z (1 - z^2)\, dz\right\}, \mathrm{Re}\left\{\int_0^z -i(1 + z^2)\, dz\right\}, \mathrm{Re}\left\{\int_0^z 2z\, dz\right\}\right) \\ &= \left(\mathrm{Re}\left\{z - \frac{1}{3}z^3\right\}, \mathrm{Re}\left\{-i\left(z + \frac{1}{3}z^3\right)\right\}, \mathrm{Re}\left\{z^2\right\}\right).\end{aligned}$$

Letting $z = u + iv$, this yields

$$\mathbf{x}(u, v) = \left(u - \frac{1}{3}u^3 + uv^2, v - \frac{1}{3}v^3 + u^2v, u^2 - v^2\right)$$

which gives the Enneper surface.

You can use the applet, *MinSurfTool*, to graph an image of this surface using the functions p and q. After opening *MinSurfTool*, make sure that the `W.E.(p,q)` tab is on top. In the appropriate boxes, put $p(z) = 1$ and $q(z) = i * z$. Then click on the **Graph** button. Remember that you can increase the size of the image of the surface by clicking on the left button on the mouse, and you can decrease the size by clicking on the right mouse button. Also, you can rotate the surface by placing the cursor arrow on the image of the surface, then click on and hold the left button on the mouse as you move the cursor.

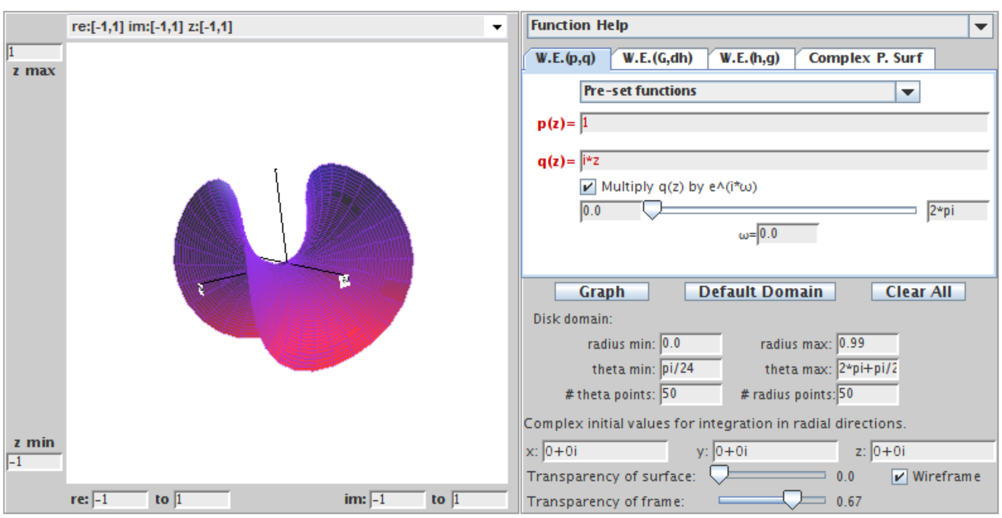


FIGURE 2.30. Enneper surface using $p(z) = 1$ and $q(z) = iz$.

EXAMPLE 2.68. Let $p(z) = 1$ and $q(z) = 1/z$ on the domain $\mathbb{C} - \{0\}$. Notice that q is meromorphic with a pole of order 1 at $z_0 = 0$ while p does not have a zero of order 2 at $z_0 = 0$. This does not violate the conditions of Theorem 2.66, because the domain is $\mathbb{C} - \{0\}$. We will show that this generates a helicoid. Using Weierstrass Representation (p,q) and letting $z = u + iv$, we get $\mathbf{x}(u, v) = (x_1, x_2, x_3)$, where

$$\begin{aligned}
x_1 &= \mathrm{Re} \int_1^z \left(1 + \frac{1}{z^2}\right) dz = \mathrm{Re}\left(z - \frac{1}{z}\right) = u - \frac{u}{u^2 + v^2} \\
x_2 &= \mathrm{Re} \int_1^z -i\left(1 - \frac{1}{z^2}\right) dz = \mathrm{Im}\left(z + \frac{1}{z}\right) = v - \frac{v}{u^2 + v^2} \\
x_3 &= \mathrm{Re} \int_1^z -2i\frac{1}{z}\, dz = 2\,\mathrm{Im}(\log z) = 2 \arg z = 2 \arctan\left(\frac{v}{u}\right).
\end{aligned}$$

Notice that this parametrization is different than the following parametrization we have been using for the helicoid:

$$\widetilde{\mathbf{x}}(\widetilde{u}, \widetilde{v}) = (\widetilde{x_1}, \widetilde{x_2}, \widetilde{x_3}) = (a \sinh \widetilde{v} \cos \widetilde{u}, a \sinh \widetilde{v} \sin \widetilde{u}, a\widetilde{u}).$$

To show that $\mathbf{x}$ also gives an image of the helicoid, we will find a substitution that will change $\mathbf{x}$ into $\widetilde{\mathbf{x}}$. Note that

$$\begin{aligned}
x_1^2 + x_2^2 &= (u^2 + v^2) - 2 + \frac{1}{u^2 + v^2} \\
\widetilde{x_1}^2 + \widetilde{x_2}^2 &= a^2 \sinh^2 \widetilde{v} = a^2 \left(\frac{e^{\widetilde{v}} - e^{-\widetilde{v}}}{2}\right)^2.
\end{aligned}$$

Equating the right hand side of these equations and letting $a = 2$, we get that

$$u^2 + v^2 = e^{2\widetilde{v}}.$$

Also, with $x_3 = \widetilde{x_3}$, we see that

$$\frac{v}{u} = \tan \widetilde{u}.$$

Now, using these last two equations we can solve for u and v to get

$$u = e^{\widetilde{v}} \cos \widetilde{u} \quad \text{and} \quad v = e^{\widetilde{v}} \sin \widetilde{u}.$$

If we substitute these values for u and v into $\mathbf{x}(u, v)$ we get the parametrization $\widetilde{\mathbf{x}}(\widetilde{u}, \widetilde{v})$ for the helicoid.

Using the `W.E.(p,q)` tab in *MinSurfTool*, we can get a graph of the helicoid by choosing from the **Pre-set functions** the values $p(z) = 1$ and $q(z) = 1/z$. Note that since the domain is $\mathbb{C} \setminus \{0\}$, the `Disk domain:  radius min:` box is set to 0.2. Also, because of the singularity at $z = 0$, there needs to be a function entered into the x, y and z boxes in the `Complex initial values for integration in radial direction` section. These functions come from explicitly solving the integrals for $x_1(z)$, $x_2(z)$, and $x_3(z)$ in the Weierstrass representation when $p(z) = 1$ and $q(z) = 1/z$ and then substituting in $re^{i\theta}$ for z. If you use a choice from the **Pre-set functions**, these functions will be entered automatically. However, if you enter your own p and q values, you will need to compute these functions and enter them into these boxes in order to get the correct minimal surface image.

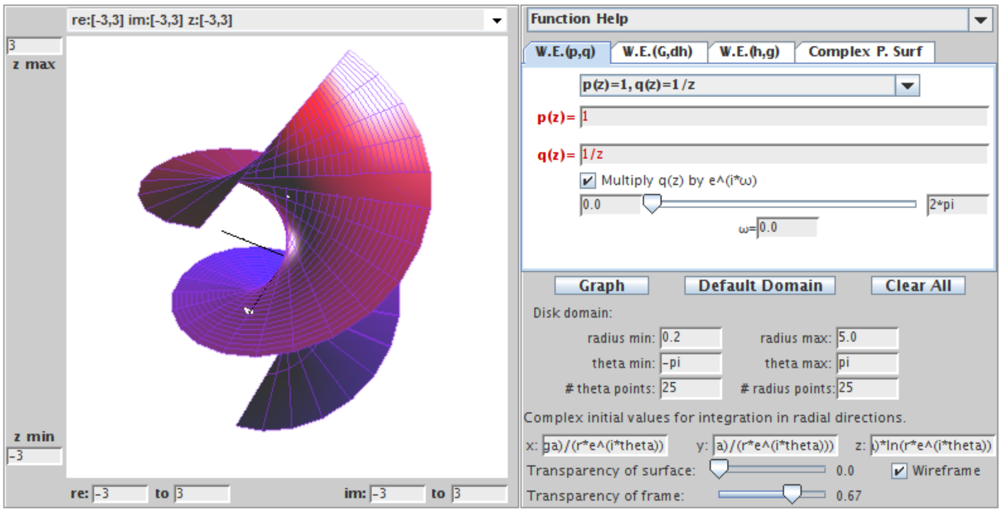


FIGURE 2.31. The helicoid using $p(z) = 1$ and $q(z) = \frac{1}{z}$.

EXERCISE 2.69. Show that the minimal surfaces generated by using $p(z) = 1$ and $q(z) = 0$ on the domain $\mathbb{C}$ in the Weierstrass representation is the plane.

Try it out!

EXERCISE 2.70. Show that the minimal surfaces generated by using $p(z) = 1$ and $q(z) = i/z$ on the domain $\mathbb{C} - \{0\}$ in the Weierstrass representation is the catenoid. Use the appropriate **Pre-set function** in the `W.E.(p,q)` tab of *MinSurfTool* to graph an image of this surface.

Try it out!

EXPLORATION 2.71. Enneper's surface can be constructed also with $p(z) = 1$ and $q(z) = z$. Recall that it has four leaves (two pointing up and two pointing down). The number of leaves can be increased.

(a) Using $p(z) = 1$ and $q(z) = z^2$ on the domain $\mathbb{C}$ in the Weierstrass representation gives the Enneper surface with six leaves (see Figure 2.32). Compute the parametrization $\mathbf{x}(u, v)$ for this surface.
(b) Conjecture the values of p and q for the Enneper surface with n leaves.
(c) Use *MinSurfTool* with the `W.E.(p,q)` tab to check your conjectured values of p and q for the Enneper surface with n leaves.

Try it out!

EXPLORATION 2.72. Use *MinSurfTool* with the `W.E.(p,q)` tab to graph an image of the surface generated by the **Pre-set functions** $p(z) = \frac{1}{1-z^4}$ and $q(z) = z$.

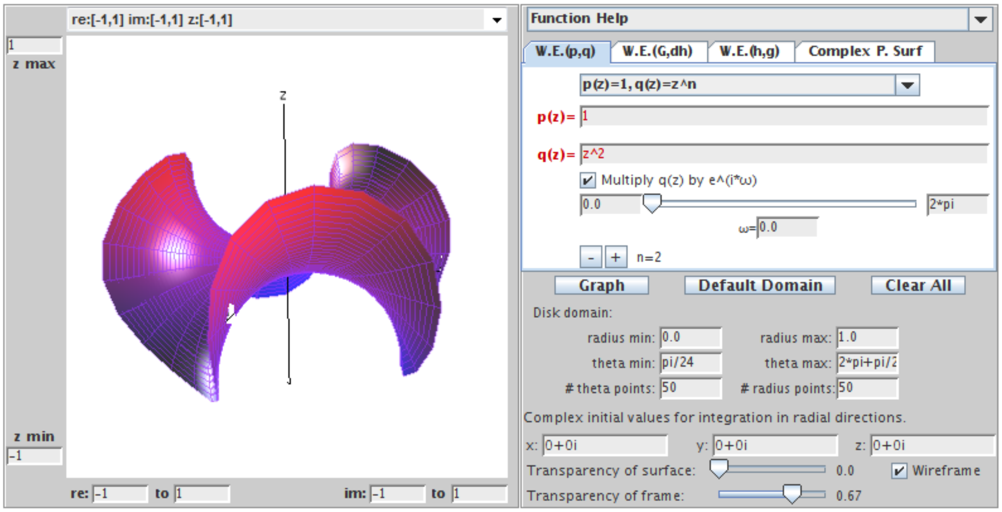


FIGURE 2.32. Enneper surface with 6 leaves using $p(z) = 1$ and $q(z) = z^2$.

(a) What minimal surface is this?
(b) Click on the box "Multiply $q(z)$ by e∧(i*ω)" and move the slider to generate a family of minimal surfaces. These surfaces are associated surfaces (see the paragraph after Definition 2.50). When $\theta = \frac{\pi}{2}$ you get the conjugate surface. In this case, what is the conjugate surface?
(c) Experiment with *MinSurfTool* to view the associated family and find the conjugate surface of various minimal surfaces discussed above.

Try it out!

EXPLORATION 2.73. Scherk's doubly periodic surface is generated with $p(z) = \frac{1}{1-z^4}$ and $q(z) = iz$. Use the appropriate **Pre-set function** in the `W.E.(p,q)` tab of *MinSurfTool* to graph an image of the surface generated by $p(z) = \frac{1}{1-z^{2n}}$ and $q(z) = iz^{n-1}$ for various values of $n = 2, 3, 4, \ldots$. Note the $-$ and $+$ boxes above the **Graph** button; these can be used to increase or decrease the value of n.

(a) What happens to the surface as n increases?
(b) Notice that the surface has leaves that alternate between going up and going down. How is n related to the number of leaves?
(c) What is the image of the projection of the surface onto the x_1x_2-plane for each n?
(d) Using the previous parts conjecture how many leaves the surface would have if $p(z) = \frac{1}{1-z^5}$. Why could such a surface not exist?

Try it out!

EXPLORATION 2.74. Use the appropriate **Pre-set function** in the `W.E.(p,q)` tab of *MinSurfTool* to graph an image of the surface generated by $p(z) = \frac{1}{(1-z^4)^2}$ and $q(z) = iz^3$. This surface is known as the 4-noid (see Figure 2.33).

(a) Try to create a 3-noid by changing the values of p and q and graphing the result in *MinSurfTool*.
(b) Conjecture the values of p and q that will generate an n-noid.

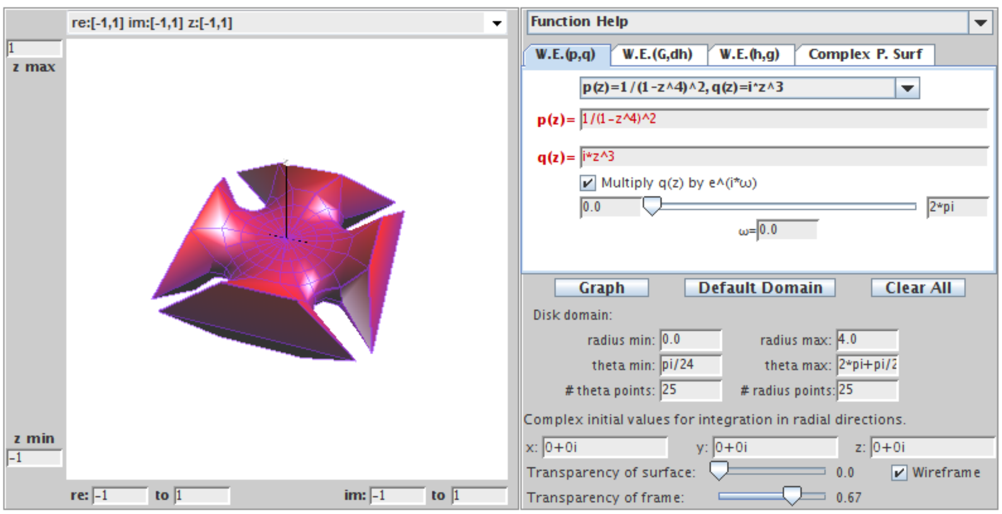


FIGURE 2.33. Image of the 4-noid minimal surface.

Try it out!

While the Weierstrass Representation will generate a minimal surface, there is no guarantee that the minimal surface will be embedded. Recall from the introduction of this chapter that we said that a surface is embedded if it has no self-intersections. The plane, the catenoid, the helicoid, and Scherk's doubly periodic surface are examples of embedded minimal surfaces. However, Enneper's surface is not embedded. In Exploration 2.11 you saw that Enneper's surface intersects itself when the domain contains a disk centered at the origin of radius $R \geq \sqrt{3}$.

EXPLORATION 2.75. Using *MinSurfTool* with the `W.E.(p,q)` tab, come up with three sets of functions p and q that create other minimal surfaces that are not embedded. Remember if the chosen p and q functions result in a singularity in the Weierstrass representation, then you will need to enter functions into the x, y and z boxes in the `Complex initial values for integration in radial direction` section. These functions come from explicitly solving the integrals for $x_1(z)$, $x_2(z)$, and $x_3(z)$ in the Weierstrass representation for the chosen $p(z)$ and $q(z)$ values and then substituting in $re^{i\theta}$ for z.

Try it out!

An important area in minimal surface theory is the study of complete (boundaryless) embedded minimal surfaces. The following theorem tells us that any minimal surface without boundary cannot be closed and bounded.

THEOREM 2.76. If M is a complete minimal surface in $\mathbb{R}^3$, then M is not compact.

PROOF. By Theorem 2.45, we can assume that M has an isothermal parametrization. Now, if M were compact, then each coordinate function would attain a maximum. Since the real part of an analytic function is harmonic we see from Theorem 2.65, the coordinate functions in this parametrization are harmonic. But harmonic functions attain their maximum on the boundary of the set. So, M must have a boundary which contradicts M being complete. □

2.5. The Gauss map, G, and height differential, dh

We can use other representations for $\phi = (\varphi_1, \varphi_2, \varphi_3)$ to form different Weierstrass representations as long as $\phi^2 = 0$ and $|\phi|^2 \neq 0$ (see the Summary on page 146). An important representation employs the functions known as the Gauss map, G, and the height differential, dh. This representation is useful, because the functions G and dh describe the geometry of the minimal surface. To develop this representation, we first need some background about the Gauss map.

Recall that the curvature of a unit speed curve, α, at a point s is $\left|\alpha''(s)\right|$. That is, the curvature of a curve is described by the rate of change of the tangent vector. Similarly, the curvature of a surface is related to the rate of change of the tangent plane. Since each tangent plane is essentially determined by its unit normal vector, $\mathbf{n}$, we can investigate the curvature of a surface by studying the variation of the unit normal vector. This is the idea behind the Gauss map.

DEFINITION 2.77. Let $M : \Omega \to \mathbb{R}^3$ be a surface with a chosen orientation (that is, a differentiable field of unit normal vectors $\mathbf{n}$). The *Gauss map*, $\mathbf{n}_p$, translates the unit normal on M at a point p to the unit vector at the origin pointing in the same direction as the unit normal and thus corresponds to a point on the unit sphere S^2.

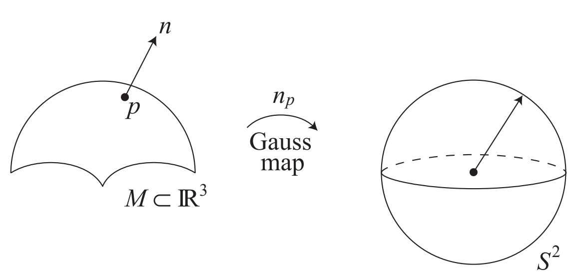


FIGURE 2.34. The Gauss map.

EXAMPLE 2.78. Let's determine the image of the Gauss map for the catenoid. A meridian is a curve formed by a vertical slice on the surface (see Exercise 2.15). Consider a meridian on the entire catenoid (remember that the image in Figure 2.35 is just part of a catenoid and that it actually extends on forever). The Gauss map, $\mathbf{n_p}$, of this meridian will be a meridian on S^2 from the north pole, $(0,0,1)$, to the south pole, $(0,0,-1)$, that excludes these end points. Note that $(0,0,1)$ and $(0,0,-1)$ are excluded, because no matter how far the catenoid extends, the unit normal $\mathbf{n}$ never points exactly straight up or exactly straight down. Now, since the catenoid is a surface of revolution if we revolve this meridian on S^2, we get that the image of the Gauss map for the catenoid is $S^2 \setminus \{(0,0,1),(0,0,-1)\}$.

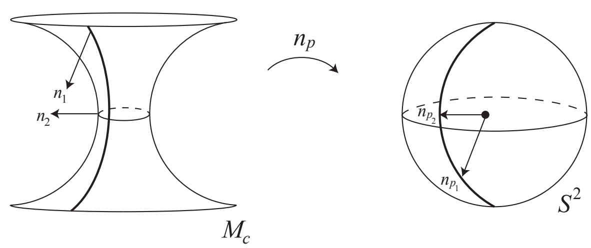


FIGURE 2.35. Image of a meridian on a catenoid under the Gauss map.

EXERCISE 2.79. Describe the image of the Gauss map for the following surfaces:

(a) a right circular cylinder;
(b) a torus;
(c) Ennepers surface defined just on $\mathbb{D}$;
(d) helicoid;
(e) Scherk's doubly periodic surface.

Try it out!

THEOREM 2.80 ([**22**]). Let M be a minimal surface with an isothermal parametrization. Then the Gauss map of M preserves angles.

While the Gauss map preserves angles, it reverses orientation. Such maps are known as *anticonformal*. To help visualize the fact that the Gauss maps reverses orientation, consider three points A, B, and C on a curved path near the neck of the catenoid (see Figure 2.36). Since A is above the neck of the catenoid, the outward pointing unit normal at A will be pointing downward and hence the Gauss map will put it below the equator on S^2 at the point A'. The point B is on the neck of the catenoid and so the outward pointing unit normal at B will be horizontal. So, the Gauss map will put it on the equator of S^2 at the point B'. Similarly, the normal at C will get mapped to C'. Thus, following the curve path from A to B to C in the positive direction on the catenoid gets sent by the Gauss map to a curve from A' to B' to C' in the negative direction on S^2. That is, we have an orientation-reversing map.

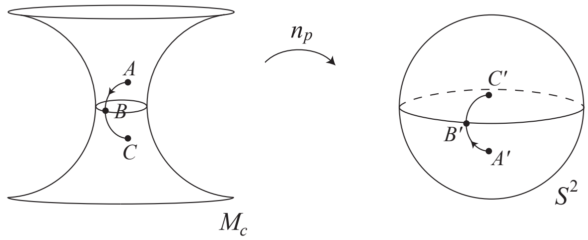


FIGURE 2.36. $\mathbf{n_p}$ is orientation reversing.

Since the Gauss map associates a point on M with a point on S^2, we can also associate it with a point in the complex plane $\mathbb{C}$ by using stereographic projection. Recall that stereographic projection, σ, takes a point on S^2 to a point in the extended complex plane, $\mathbb{C} \cup \infty$. To do this, we place the complex plane through the equator of the sphere and take a line connecting the north pole, $(0, 0, 1) \in S^2$, with the given point $(x_1, x_2, x_3) \in S^2$. This line will intersect the extended complex plane at some point, $z = x + iy$. In such a setting the unit sphere is known as the *Riemann sphere*.

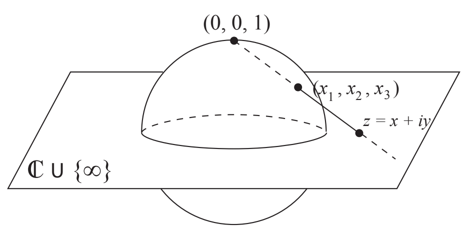


FIGURE 2.37. Stereographic projection.

EXERCISE 2.81. Describe the projections of the following sets on the Riemann sphere onto the extended complex plane:

(a) meridians;
(b) parallels;
(c) circles;
(d) circles that contain (i.e., touch) the point $(0, 0, 1)$;
(e) antipodal points (i.e., diametrically opposite points).

Try it out!

Finally, let $\overline{\sigma}$ be the projection of $(x_1, x_2, x_3) \in S^2$ to the point $x - iy \in \mathbb{C}$ given first by stereographic projection of (x_1, x_2, x_3) to $z = x + iy$ followed by reflection of $z = x + iy$ across the real axis to $\overline{z} = x - iy$. Note that $\overline{\sigma}$ is anticonformal.

Now, let $G : D \subset \mathbb{C} \to \mathbb{C}$ be the map defined by $G = \overline{\sigma} \circ \mathbf{n} \circ \mathbf{x}$. Note that G preserves angles since $\overline{\sigma}$, $\mathbf{n}$, and $\mathbf{x}$ preserves angles and so the composition preserves angles. Also,

G is orientation preserving, because both $\overline{\sigma}$ and $\mathbf{n}$ are orientation reversing and so their composition is orientation preserving. Thus, G is a meromorphic function. It is also called the Gauss map. The map G is of more interest to us, so for the remainder of the chapter "the Gauss map" will refer to this map.

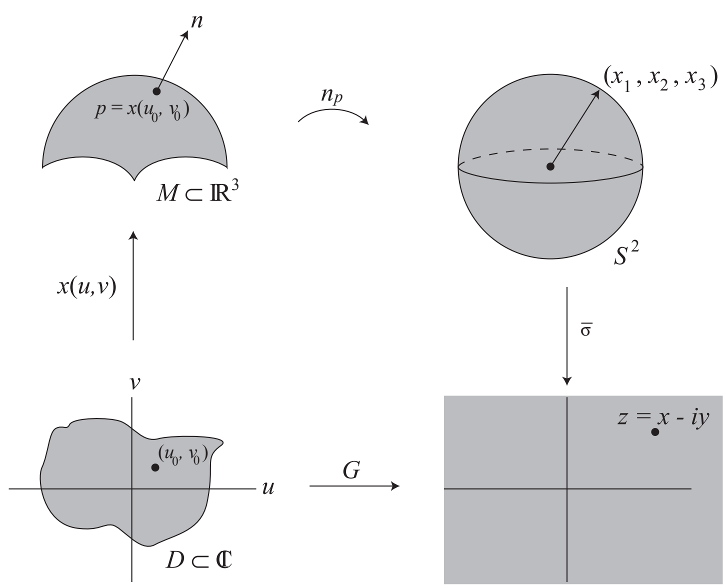


FIGURE 2.38. The map G.

EXAMPLE 2.82. Using the geometry of the Enneper surface, M_E, we can determine specific values of a Gauss map, G, on M_E even though we do not know what function G is. What is $G(0)$? Enneper's surface is formed by bending a disk into a saddle surface. The point $0 \in \mathbb{C}$ should get mapped to the point in the center of the Enneper surface. For simplicity sake, we will take the downward pointing normal $\mathbf{n}$. Hence, the unit normal at the center of M_E points straight down, and thus mapping it to S^2 under $\mathbf{n_p}$ gives the vector pointing at $(0, 0, -1)$ (see Figure 2.40). Taking the stereographic projection, σ, results in the point $z = 0 \in \mathbb{C}$ and reflecting this across the real axis does not change 0, so $\overline{\sigma}(0, 0, -1) = 0$. Hence, $G(0) = 0$.

Next, what is $G(r)$ when $r \in [0, 1]$? The points r get mapped under $\mathbf{x}$ to a curve moving upward along one of the upward pointing leaves of the Enneper surface. The corresponding downward pointing unit normal, n_r, stays in the x_1x_3-half plane (where $x_1 \geq 0$) also moving upward (i.e., the x_3 value is increasing). As r approaches 1, n_r approaches being parallel to the x_1 axis (see Figure 2.41). Thus, mapping these unit normals to S^2, the curve $\{r \in \mathbb{D} : 0 \leq r < 1\}$ traces a meridian on S^2 from $(0, 0, -1)$ to $(1, 0, 0)$. The stereographic projection of this onto the complex plane gives $\{r \in \mathbb{D} : 0 \leq r < 1\}$ and reflecting this across the real axis does not change the values. Hence, $G(r) = r$, where $0 \leq r \leq 1$.

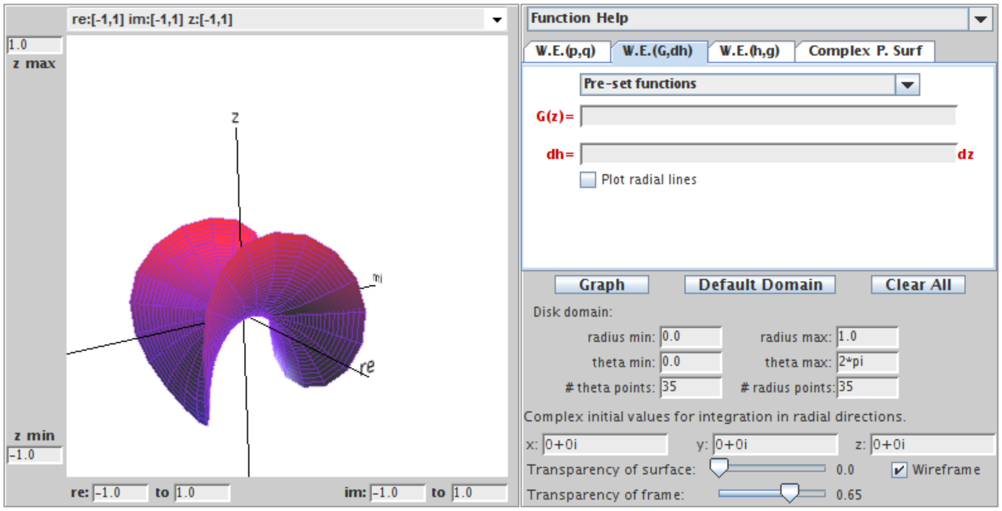


FIGURE 2.39. The Enneper surface.

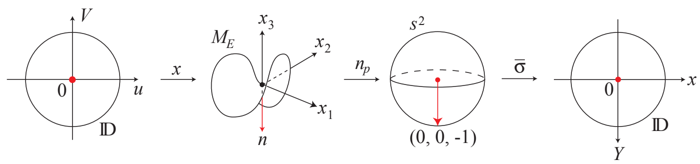


FIGURE 2.40. $G(0) = 0$ for the Enneper surface.

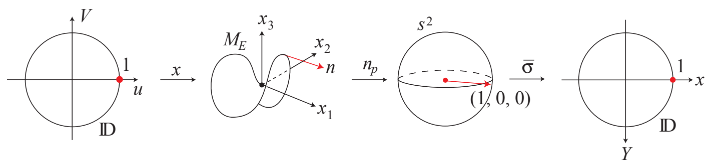


FIGURE 2.41. $G(1) = 1$ for the Enneper surface.

Finally, what is $G(e^{i\theta})$ for $0 \leq \theta \leq \frac{\pi}{2}$? If we restrict the domain of the Enneper surface to $\mathbb{D}$, these points get mapped to the edge of our image of the Enneper surface in a positive direction. At $\theta = 0$, the unit normal is pointing outward (i.e., away from the opposite leaf) and under the Gauss map, n_p, this corresponds to $(1, 0, 0) \in S^2$. As θ moves from 0 to $\frac{\pi}{2}$, the unit normal moves from pointing outward to pointing inward (i.e., toward the opposite leaf). So that at $\theta = \frac{\pi}{2}$, the unit normal is mapped under n_p to $(0, -1, 0) \in S^2$ which projects under σ to $-i \in \mathbb{C}$ (see Figure 2.42). Reflecting this across the real axis gives $\overline{\sigma}(0, -1, 0) = \overline{-i} = i$. A similar argument shows that the same thing happens for all $\theta \in [0, \frac{\pi}{2}]$. That is, $G(e^{i\theta}) = e^{i\theta}, (0 \leq \theta \leq \frac{\pi}{2})$.

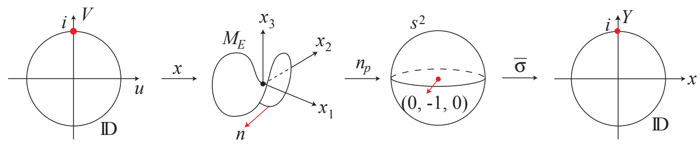


FIGURE 2.42. $G(i) = i$ for the Enneper surface.

EXAMPLE 2.83. Let's determine some specific values for the Gauss map G for the singly periodic Scherk surface with six leaves, M_S. The domain used here is $\mathbb{D}$. Also, the leaves are centered at rays from the origin through each of the 6th roots of unity (i.e., $e^{i\pi k/3}, (k = 0, ..., 5)$) with the leaf centered at the positive real axis pointing upward and the subsequent leaves alternating between downward pointing and upward pointing (see Figure 2.43).

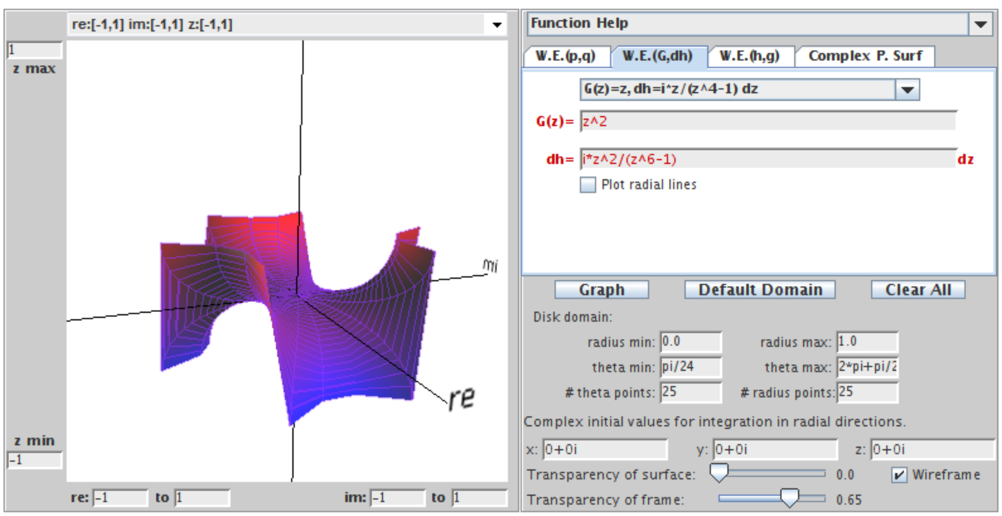


FIGURE 2.43. The singly periodic Scherk surface with six leaves.

As in the previous example we will use the downward pointing unit normal, and so $G(0) = 0$. Next, because the leaves are centered at the 6th roots of unity (i.e., $e^{i\pi k/3}, (k = 0, ..., 5)$), let's look at $G(e^{i\pi k/3})$, where $(k = 0, ..., 5)$. First, the point 1 gets mapped under $\mathbf{x}$ to the "edge" of M_S above the positive real axis. By looking at the graph of M_S, we see that this is in the middle of an upward pointing leaf, and the corresponding unit normal $\mathbf{n}$ lies above the positive real axis and pointing away from the origin. Also, it lies in a plane parallel to the horizontal x_1x_2-plane. Mapping this normal under $\mathbf{n}_p$ and then $\overline{\sigma}$, results in the point 1. Hence, $G(1) = 1$. Now, consider $G(e^{i\pi/3})$. The point $e^{i\pi/3}$ gets mapped to the "edge" of M_S above the line $re^{i\pi/3}, r > 0$. Since the leaves alternate between pointing upward and pointing downward, this is in the middle of a downward pointing leaf and points toward the origin. Mapping this normal under $\mathbf{n}_p$ and then $\overline{\sigma}$, results in the point $\overline{e^{i4\pi/3}} = e^{i2\pi/3}$. Hence, $G(e^{i\pi/3}) = e^{i2\pi/3}$.

In a similar way, we get the following values:

$$G(1) = 1, \qquad G\left(e^{i\frac{\pi}{3}}\right) = e^{i\frac{2\pi}{3}}, \qquad G\left(e^{i\frac{2\pi}{3}}\right) = e^{i\frac{4\pi}{3}},$$

$$G(-1) = 1, \qquad G\left(e^{i\frac{4\pi}{3}}\right) = e^{i\frac{2\pi}{3}} = e^{i\frac{8\pi}{3}}, \qquad G\left(e^{i\frac{5\pi}{3}}\right) = e^{i\frac{4\pi}{3}} = e^{i\frac{10\pi}{3}}.$$

EXERCISE 2.84. A picture of the half catenoid on its side defined on $\mathbb{D}$ is shown in Figure 2.44 with the positive real axis on the right.

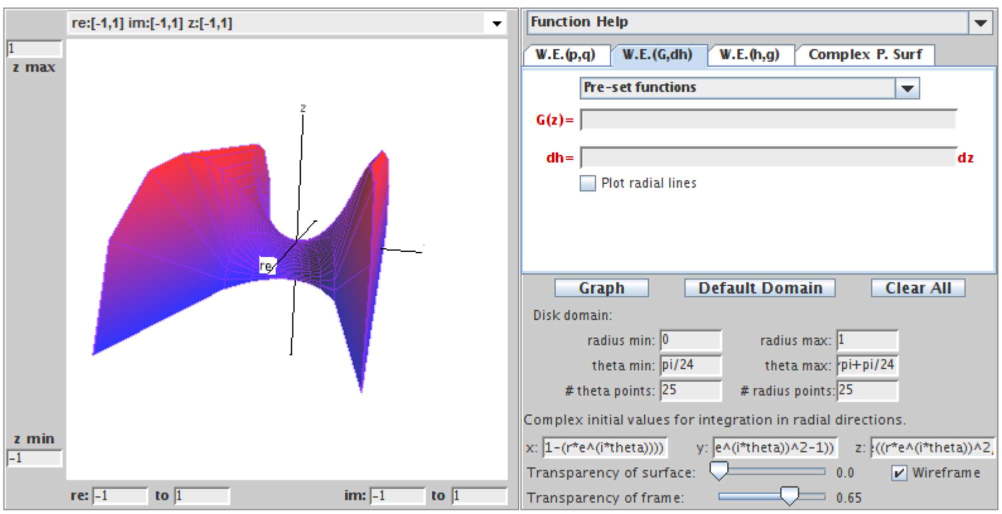


FIGURE 2.44. A view of the half catenoid on its side.

For this catenoid on its side determine:

(a) $G(0)$; (b) $G(1)$; (c) $G(-1)$; (d) $G(i)$; (e) $G(-i)$.

Try it out!

EXERCISE 2.85. For the 4-noid (see Figure 2.33), determine:

(a) $G(0)$; (b) $G(1)$; (c) $G(-1)$; (d) $G(i)$; (e) $G(-i)$.

Try it out!

We will use the Gauss map, G, to form another Weierstrass representation of a parametrized minimal surface. In doing so, we will also need the height differential,

dh, which is called such because it is locally (though not globally) the differential of the height coordinate. We will not get into the definition of differential forms; if you are interested in learning about differential forms, check out [**26**]. However, it is worth mentioning that at points where the Gauss map is vertical (i.e., $G = 0$ or $G = \infty$), the height function ought to have local minimums and maximums. Hence, dh ought to have a zero at these points (for example, see Figure 2.40).

THEOREM 2.86 (Weierstrass Representation (G,dh)). Every regular minimal surface has a local isothermal parametric representation of the form

$$\mathbf{x} = \mathrm{Re} \int_a^z \left(\frac{1}{2}\left(\frac{1}{G} - G\right), \frac{i}{2}\left(\frac{1}{G} + G\right), 1 \right) dh, \tag{14}$$

where G is the Gauss map, dh is the height differential, and $a \in \Omega$ is a constant.

PROOF. From the Summary on page 146, we need

$$\phi^2 = 0 \quad \text{and} \quad |\phi|^2 \neq 0 \text{ and be finite.}$$

Comparing eq (14) and eq (13), we have that

$$\varphi_1\, dz = \frac{1}{2}\left(\frac{1}{G} - G\right) dh, \quad \varphi_2\, dz = \frac{i}{2}\left(\frac{1}{G} + G\right) dh, \quad \varphi_3\, dz = dh.$$

In Exercise 2.87 you will show that

$$\phi^2 = 0 \qquad \text{and} \qquad |\phi|^2 \neq 0.$$

$\square$

EXERCISE 2.87. Prove that $\phi^2 = 0$ and $|\phi|^2 \neq 0$ in the proof of Theorem 2.86.
Try it out!

Note that

$$G = \frac{\varphi_1 + i\varphi_2}{-\varphi_3} \qquad \text{and} \qquad dh = \varphi_3\, dz.$$

One advantage of using this Weierstrass representation with the Gauss map and height differential is that the complex analytic properties of G and dh are related to the geometry of a minimal surface. We will discuss this in a bit, but first we will look at some examples. The following is a list of the Weierstrass data for some common minimal surfaces.

(a) The Enneper surface:	$G(z) = z$	$dh = z\, dz$	on $\mathbb{C}$.
(b) The catenoid:	$G(z) = z$	$dh = \frac{1}{z}\, dz$	on $\mathbb{C} \setminus \{0\}$.
(c) The helicoid:	$G(z) = z$	$dh = \frac{i}{z}\, dz$	on $\mathbb{C} \setminus \{0\}$.
(d) Scherk's doubly periodic surface:	$G(z) = z$	$dh = \frac{z}{z^4-1}\, dz$	on $\mathbb{D}$.
(e) Scherk's singly periodic surface:	$G(z) = z$	$dh = \frac{iz}{z^4-1}\, dz$	on $\mathbb{D}$.
(f) Polynomial Enneper:	$G(z) = p(z)$	$dh = p(z)\, dz$	on $\mathbb{C}$.
(g) Wavy plane:	$G(z) = z$	$dh = dz$	on $\mathbb{C} \setminus \{0\}$.

EXAMPLE 2.88. For $G(z) = z^k$ and $dh = z^k\,dz$, where $k = 1, 2, \ldots$, we get

$$\begin{aligned}\mathbf{x} &= \operatorname{Re}\int_0^z \left(\frac{1}{2}\left(\frac{1}{z^k} - z^k\right), \frac{i}{2}\left(\frac{1}{z^k} + z^k\right), 1\right) z^k\,dh\\ &= \left(\operatorname{Re}\frac{1}{2}\left\{z - \frac{1}{2k+1}z^{2k+1}\right\}, \operatorname{Re}\frac{1}{2}\left\{-i\left(z + \frac{1}{2k+1}z^{2k+1}\right)\right\}, \operatorname{Re}\left\{\frac{z^{k+1}}{k+1}\right\}\right).\end{aligned}$$

This is the Enneper surface with $2k+2$ leaves (see Exploration 2.71).

EXERCISE 2.89. You may have noticed that the Weierstrass data for the catenoid and the helicoid, which are conjugate surfaces (see Definition 2.50), have the same Gauss map, G, while the height diffferentials dh differ by a multiple of i. Prove that this is true for any conjugate surfaces.

Try it out!

EXERCISE 2.90. Let $G(z) = z^4$ and $dh = z^2\,dz$.

(a) Using eq (14), compute the parametrization.
(b) This minimal surface has a planar end (i.e., looks like a plane) and an Enneper end. To graph this surface, use the corresponding entry under the **Pre-set functions** in the **W.E. (G,dh)** tab of *MinSurfTool*.

Try it out!

The catenoid and the surface in Exercise 2.90 are examples of minimal surfaces with ends. Loosely, an *end* of a minimal surface is a piece that "goes on forever," or, more precisely, leaves all compact subsets of the minimal surface. Recall from Theorem 2.76 that all complete minimal surfaces in $\mathbb{R}^3$ are not compact, and hence they must possess at least one end.

EXERCISE 2.91. Determine the number of ends each of the following surfaces have:
(a) the catenoid; (b) the plane; (c) the helicoid; (d) Enneper's surface.

Try it out!

Ends occur in a deleted neighborhood (i.e., a disk with the centered removed) centered at a singularity. Three common types of ends for minimal surfaces are: (1) Enneper ends; (2) catenoid ends; and (3) flat or planar ends. In discussing ends, we will need to represent ds, the metric (i.e., a way to measure distance) on a minimal surface, in terms of G and dh. Using $ds^2 = |\phi|^2$ and eq (14), we derive

$$ds = \frac{1}{\sqrt{2}}\left(|G| + \frac{1}{|G|}\right)|dh|. \tag{15}$$

An Enneper end has $ds \sim |z^k| \cdot |dz|$, so the metric grows in the same manner as a polynomial as we approach the end. On the other hand, a catenoidal end and a planar end have $ds \sim |dz|$, so the metric becomes Euclidean. A catenoidal end differs from a planar end in that the residue of dh is logarithmic.

Exercise 2.92. Prove eq. (15).
Try it out!

Example 2.93. We know that the catenoid has two catenoid ends, but let's show how we could prove this if we did not know what type of ends these are. Using the Weierstrass data, $G(z) = z$ and $dh = \frac{1}{z}\,dz$, for the catenoid, we have the parametrization

$$\mathbf{x}(z) = \left(\frac{1}{2}\operatorname{Re}\left(-\frac{1}{z} - z\right), \frac{i}{2}\operatorname{Re}\left(-\frac{1}{z} + z\right), \operatorname{Re}\left(\log z\right)\right).$$

So there is a singularity or pole of order 1 at 0. Also, there is a pole of order 1 at ∞. To see that there is a singularity at ∞, we replace z with $\frac{1}{w}$ and look at the limit as w goes to 0. Thus, the catenoid will have two ends (one at 0 and one at ∞). To determine what types of ends these are, we look at ds at these points. Note that

$$ds = \frac{1}{\sqrt{2}}\left(|z| + \frac{1}{|z|}\right)\frac{1}{|z|}|dz|.$$

As $z \to \infty$, $ds \sim |dz|$ and because x_3 is logarithmic, we have a catenoid end.

At $z = 0$, plugging in 0 does not work, so instead we let $w = \frac{1}{z}$ and consider $w \to \infty$. Note that in this case,

$$dh = \frac{1}{z}\,dz = w\left(\frac{1}{w}\right)' = w\left(-\frac{1}{w^2}\,dw\right) = -\frac{1}{w}\,dw.$$

Therefore,

$$ds = \frac{1}{\sqrt{2}}\left(|w| + \frac{1}{|w|}\right)\frac{1}{|w|}|dw|.$$

As $w \to \infty$, $ds \sim |dw|$ and again because x_3 is logarithmic, we have a catenoid end.

Example 2.94. The surface in Exercise 2.90 has $G(z) = z^4$ and $dh = z^2\,dz$, and the corresponding parametrization is

$$\mathbf{x}(z) = \left(\frac{1}{2}\operatorname{Re}\left(\frac{1}{7}z^7 - \frac{1}{z}\right), \frac{i}{2}\operatorname{Re}\left(\frac{1}{7}z^7 + \frac{1}{z}\right), \operatorname{Re}\left(\frac{1}{3}z^3\right)\right).$$

Note there are singularities at 0 and at ∞ and

$$ds = \frac{1}{\sqrt{2}}\left(|z|^4 + \frac{1}{|z|^4}\right)|z|^2|dz|.$$

As $z \to \infty$, $ds \sim |z|^6\,|dz|$ and so we have an Enneper end.

At $z = 0$, we again let $w = \frac{1}{z}$ and consider $w \to \infty$. Then $G(w) = w^4$ and $dh = -\frac{1}{w^4}\,dw$. Hence,

$$ds = \left(|w|^4 + \frac{1}{|w|^4}\right)\frac{1}{|w|^4}|dw|.$$

As $w \to \infty$, $ds \sim |dw|$, but because there is no logarithmic term, we have a planar end.

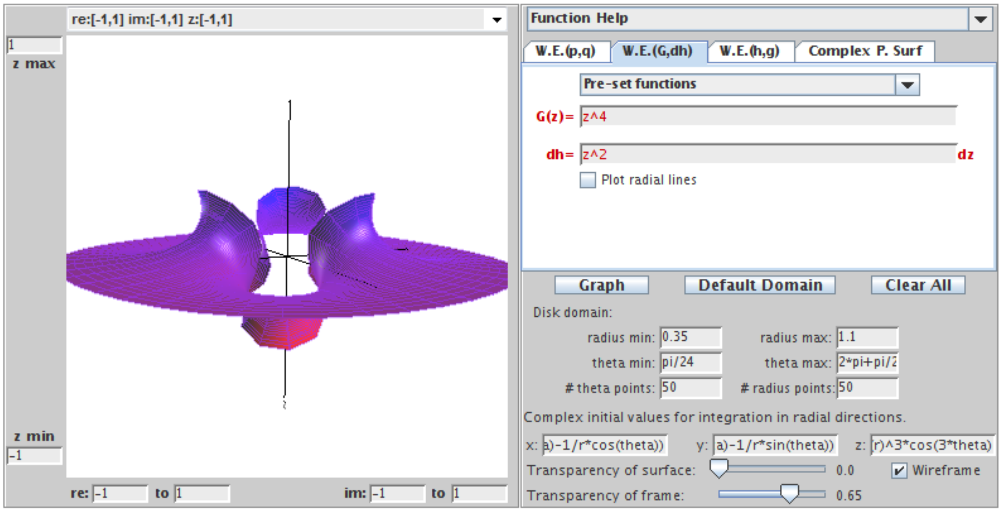


FIGURE 2.45. A minimal surface with Enneper and planar ends.

EXERCISE 2.95. Let $G(z) = \frac{z^2+3}{z^2-1}$ and $dh = \frac{z^2+3}{z^2-1}\, dz$. Show that this minimal surface has one planar end and two catenoid ends.

Try it out!

The Gauss map and height differential also tell us about two important types of curves on a minimal surface. These are known as the asymptotic lines and the curvature lines. To understand what these lines are, we will review the terms normal curvature and principal directions, both of which were discussed in Section 2. Let p be a point on a curve on a minimal surface M. The tangent vector $\mathbf{w}$ and normal vector $\mathbf{n}$ at p form a plane that intersects the surface in another curve, say α (see Figure 2.18). The normal curvature in the direction $\mathbf{w}$ is $\alpha'' \cdot \mathbf{n}$ and measures how much the surface bends toward $\mathbf{n}$ as you move in the direction of $\mathbf{w}$ at point p. An *asymptotic line* is a curve that is tangent to a direction in which the normal curvature is zero.

As we rotate the plane through the normal $\mathbf{n}$, we will get a set of curves on the surface each of which has a value for its curvature. The directions in which the normal curvature attains its absolute maximum and absolute minimum values are known as the principal directions. *Curvature lines* are curves that are always tangent to a principal direction.

A nice relationship between these lines and the Weierstrass data is:

$$\text{A curve } z(t) \text{ is an asymptotic line} \iff \frac{dG}{G}(z) \cdot dh(z) \in i\mathbb{R}.$$

$$\text{A curve } z(t) \text{ is a curvature line} \iff \frac{dG}{G}(z) \cdot dh(z) \in \mathbb{R}.$$

Example 2.96. Let $G(z) = z$ and $dh(z) = dz$. This is a parametrization of the wavy plane. Computing the Weierstrass representation, we get the parametrization:

$$(x_1(z), x_2(z), x_3(z)) = \left(\operatorname{Re}\left\{ \frac{1}{2}\log(z) - \frac{1}{4}z^2 \right\}, \operatorname{Re}\left\{ \frac{i}{2}\log(z) + \frac{i}{4}z^2 \right\}, \operatorname{Re}\left\{ z \right\} \right).$$

We can plot an image of this surface using the corresponding function under **Pre-set functions** in the **W.E.(G,dh)** tab in *MinSurfTool*.

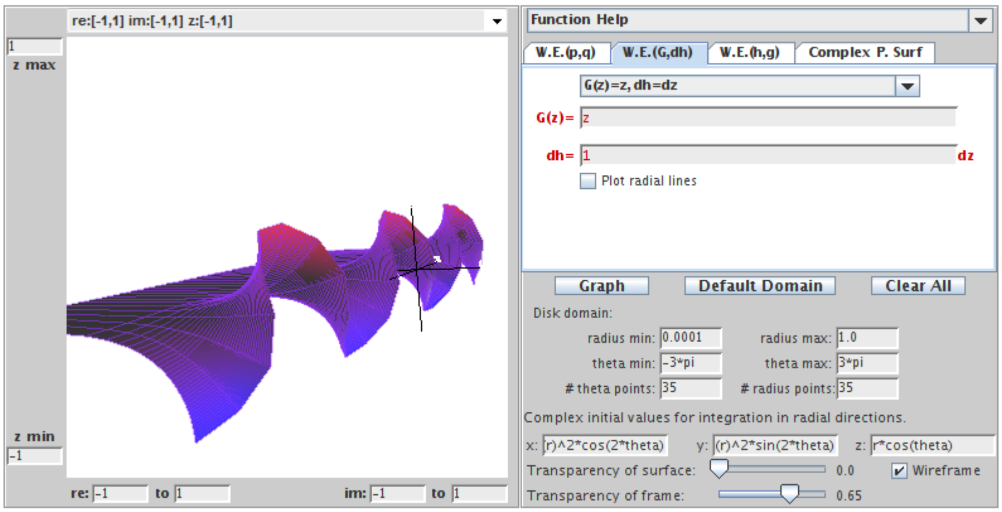


Figure 2.46. The wavy plane.

Now, for the wavy plane

$$\frac{dG}{G}(z) \cdot dh(z) = \frac{dz}{z} \cdot dz.$$

If we let $z = e^{i\theta}$ (since we let radius max=1), then $dz = ie^{i\theta}d\theta$ and

$$\left. \frac{dG}{G}(z) \cdot dh(z) \right|_{z=e^{i\theta}} = -e^{i\theta}(d\theta)^2.$$

So, from the equations above, we get that for $k \in \mathbb{Z}$: (1) the asymptotic lines occur when $\theta = \frac{\pi}{2} + k\pi$; and (2) the curvature lines occur when $\theta = k\pi$.

If we use *MinSurfTool* to plot the wavy plane with theta min $= -\frac{\pi}{2}$ and theta max $= \frac{\pi}{2}$, we see that these asymptotic lines lie in the x_1x_2-plane. Similarly, if we plot theta min $= 0$ and theta max $= \pi$, we see that these curvature lines are reflection lines through which the wavy plane can be reflected as if through a mirror to get a smooth continuation of the minimal surface.

Exercise 2.97. Using $G(z) = z$ and $dh = zdz$ for Enneper's surface, show that for $z = e^{i\theta}$ the asymptotic lines occur when $\theta = \frac{\pi}{4} + \frac{k\pi}{2}$, where $k \in \mathbb{Z}$, and the curvature lines occur when $\theta = \frac{k\pi}{2}$, where $k \in \mathbb{Z}$. Use *MinSurfTool* to plot these lines on Enneper's surface.

Try it out!

EXERCISE 2.98. Prove that for conjugate surfaces, the asymptotic lines (and curvature lines) of one surface are the curvature lines (and asymptotic lines) of the other surface.

Try it out!

As mentioned earlier, an advantage of using the Gauss map and height differential is that the complex analytic properties of G and dh are related to the geometry of a minimal surface. So, let's look at the complex analytic properties of G and dh. First, recall from the Summary on page 146, we need $|\phi|^2$ to be finite and nonzero. Because of eq (15), this leads to the following condition.

PROPOSITION 2.99. *At a nonsingular point, G has a zero or pole of order n if and only if dh has a zero of order n.*

Second, note that the integrals in the Weierstrass representation in eq (14) might depend upon the path of integration if the domain of G and dh is not simply connected. If the representation does not depend on the path of integration, then all closed paths γ in the domain satisfy the three conditions below:

$$\begin{aligned}
&\operatorname{Re}\left(\frac{1}{2}\int_\gamma \left(\frac{1}{G} - G\right)\, dh\right) = 0;\\
&\operatorname{Re}\left(\frac{i}{2}\int_\gamma \left(\frac{1}{G} + G\right)\, dh\right) = 0;\\
&\operatorname{Re}\int_\gamma dh = 0.
\end{aligned}$$

If a curve has the property that one of these quantities is a nonzero constant, then the surface will have periodic behavior. Example 2.101 illustrates an example of this behavior.

The three equations above can be reduced to the following two period conditions:

$$\begin{aligned}
&(i)\ \int_\gamma G\, dh = \overline{\int_\gamma \frac{1}{G}\, dh} \quad \text{(horizontal period condition)},\\
&(ii)\ \operatorname{Re}\int_\gamma dh = 0 \quad \text{(vertical period condition)}.
\end{aligned} \tag{16}$$

for all closed paths γ in the domain.

EXERCISE 2.100. Show that the conditions

$$\operatorname{Re}\left(\frac{1}{2}\int_\gamma \left(\frac{1}{G} - G\right)\, dh\right) = 0, \text{ and } \operatorname{Re}\left(\frac{i}{2}\int_\gamma \left(\frac{1}{G} + G\right)\, dh\right) = 0$$

are equivalent to the condition

$$\int_\gamma G\,dh = \overline{\int_\gamma \frac{1}{G}\,dh}.$$

Try it out!

These period conditions are useful in determining horizontal periods (e.g., Scherk's doubly periodic surface), vertical periods (e.g., Scherk's singly periodic surface) and in determining possible constant values in G and dh.

EXAMPLE 2.101. Consider Scherk's doubly periodic surface with the Weierstrass data $G(z) = z$ and $dh(z) = \frac{z}{z^4-1}dz$. The horizontal period condition is $\int_\gamma G\,dh = \overline{\int_\gamma \frac{1}{G}\,dh}$, for all closed paths γ in the domain. Note that both integrands are meromorphic with poles of order 1 at $\pm 1, \pm i$. So, the only paths that concern us are ones that enclose one, two, or three of these poles. A nice way to calculate these integrals along such paths is to use the Residue Theorem that states if γ is a simple closed positively-oriented contour and f is analytic inside and on γ except at the points $z_1, \ldots, z_n$ inside γ, then

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^n \operatorname{Res}(f, z_j).$$

Recall that for poles of order 1,

$$\operatorname{Res}(f, z_j) = \lim_{z\to z_j} (z - z_j) f(z).$$

Thus, for $\displaystyle\int_\gamma G\,dh$, we have

$$\operatorname{Res}(G\,dh, z_j) = \lim_{z\to z_j} \frac{z^3 - z_j z^2}{z^4-1} = \lim_{z\to z_j} \frac{3z^2 - 2z_j z}{4z^3} = \lim_{z\to z_j} \frac{3z^3 - 2z_j z^2}{4z^4} = \frac{z_j^3}{4}.$$

In particular,

$$\operatorname{Res}(G\,dh, 1) = \frac{1}{4} \qquad \operatorname{Res}(G\,dh, i) = \frac{-i}{4}$$
$$\operatorname{Res}(G\,dh, -1) = \frac{-1}{4} \qquad \operatorname{Res}(G\,dh, -i) = \frac{i}{4}.$$

Similarly, we can compute that

$$\operatorname{Res}\left(\frac{1}{G}\,dh, 1\right) = \frac{1}{4} \qquad \operatorname{Res}\left(\frac{1}{G}\,dh, i\right) = \frac{i}{4}$$
$$\operatorname{Res}\left(\frac{1}{G}\,dh, -1\right) = \frac{-1}{4} \qquad \operatorname{Res}\left(\frac{1}{G}\,dh, -i\right) = \frac{-i}{4}.$$

Now, if the path γ_1 just contains the pole at $z_1 = 1$, then the horizontal period conditions result in

$$\int_{\gamma_1} G\,dh = 2\pi i \operatorname{Res}(G\,dh, 1) = \frac{i\pi}{2}, \qquad \overline{\int_{\gamma_1} \frac{1}{G}\,dh} = \overline{2\pi i \operatorname{Res}\left(\frac{1}{G}\,dh, 1\right)} = \frac{-i\pi}{2}.$$

These integrals should be equal, which occurs if the minimal surface is periodic in the imaginary direction with period of π. Likewise, if we take a path γ_2 that just contains the pole $z_2 = i$, then we get

$$\int_{\gamma_2} G\,dh = \frac{\pi}{2}, \qquad \overline{\int_{\gamma_2} \frac{1}{G}\,dh} = \frac{-\pi}{2},$$

and the minimal surface is periodic in the real direction with period π. All other paths γ are covered by these two cases. Finally, if we look at the vertical period condition, we get that the condition is automatically true for all paths γ and so the minimal surface is not periodic in the vertical direction. This matches the image of Scherk's doubly periodic surface from *MinSurfTool*.

Exercise 2.102. Show that the period conditions given in eq. (16) result in Scherk's singly periodic surface being periodic in the vertical direction.

Try it out!

Let's look at example of how all of this can help us use the geometry of a minimal surface to determine G and dh.

Example 2.103. From the list of Weierstrass data on page 159, we know that $G(z) = z$ and $dh = z\,dz$ for the Enneper surface. However, we want to show how this Weierstrass data can be determined by using the geometric shape of the surface. First, let's determine a plausible candidate for G. To do this, we will make a guess based on the value of G at a few specific points. From Example 2.82, we know that $G(0) = 0$, $G(r) = r$ for $0 \le r \le 1$, and $G(e^{i\theta}) = e^{i\theta}$ for $0 \le \theta \le \frac{\pi}{2}$. Therefore, it seems plausible to let $G(z) = z$. Second, given this G, let's determine dh. Because eq (15) must be finite and the Enneper surface has no ends in $\mathbb{C}$, dh cannot have any poles in $\mathbb{C}$. However, from the sentence before Exercise 2.91, we know that Enneper's surface must have at least one end. This end corresponds to the point at infinity, $z = \infty$, and so dh has a pole at ∞. Thus, $dh = \rho z^n\,dz$, for some $n \in \mathbb{N}$ and $\rho \in \mathbb{C}$. Since $G(z) = z$ has a zero of order 1 at 0, by Proposition 2.99 dh must also have a zero of order 1 at 0 and no other zeros. Thus, $dh = \rho z\,dz$. For simplicity sake, we let $\rho = 1$. Finally, notice that the period conditions in eq (16) hold, because there are no poles in $\mathbb{C}$, and so every integral along any closed path γ will equal 0. Hence, the Weierstrass data

$$G(z) = z, \qquad dh = z\,dz$$

generates a minimal surface.

EXAMPLE 2.104. Consider the singly periodic Scherk surface with six leaves, M_S (see Figure 2.43). Note that these leaves go off to infinity. Hence, we will have 6 poles. Because of symmetry, we will choose these poles to be at the 6th roots of unity (i.e., $e^{i\pi k/3}, (k = 0, ..., 5)$). This means that dh will have the term $z^6 - 1$ in its denominator. However, we will need to determine G first in order to know what should be in the numerator of dh. From the results in Example 2.83, it seems reasonable that $G(z) = z^2$. Since $G(z) = z^2$ has a zero of order 2 at 0, by Proposition 2.99 dh must also have a zero of order 2 at 0 and no other zeros. Thus, we so far have

$$G(z) = z^2, \qquad dh = \rho \frac{z^2}{z^6 - 1}\, dz,$$

where $\rho \in \mathbb{C}$. To determine possible values of ρ, consider the period conditions in eq (16). There are poles of order 1 at $e^{ik\pi/3}, k = 0, \ldots, 5$. We compute that

$$\operatorname{Res}(G\, dh, z_j) = \frac{\rho z_j^5}{6}, \qquad \operatorname{Res}\left(\frac{1}{G}\, dh, z_j\right) = \frac{\rho z_j}{6}.$$

Hence, if γ contains the pole z_j, then the horizontal period condition requires

$$\int_\gamma G\, dh = 2\pi i \operatorname{Res}(G\, dh, z_j) = \frac{\rho \pi i z_j^5}{3}, \text{ and}$$

$$\overline{\int_\gamma \frac{1}{G}\, dh} = \overline{2\pi i \operatorname{Res}\left(\frac{1}{G}\, dh, z_j\right)} = \frac{-\overline{\rho}\pi i \overline{z}_j}{3}$$

to be equal (since there is no periodicity of M_S in the horizontal direction).These integrals will be equal for these poles $z_j^5 = \overline{z}_j$ if $\rho = -\overline{\rho}$. That is, ρ is purely imaginary. Without loss of generality, we let $\rho = i$ and we check that the vertical period condition holds. Hence, we have that the Weierstrass data for singly periodic Scherk surface with six leaves is

$$G(z) = z^2, \qquad dh = \frac{iz^2}{z^6 - 1}\, dz.$$

EXERCISE 2.105. Let M be the Enneper surface with 8 leaves. Using the approach of Example 2.103 determine G and dh for this surface.

Try it out!

EXERCISE 2.106. Let M be the 3-noid with ends symmetrically placed so that if the surface is rotated by $\frac{2\pi}{3}$ you will get the same image. Determine G and dh for this surface.

Try it out!

SMALL PROJECT 2.107. Let M be a minimal surface that has 6 symmetrically-placed catenoidal ends with 4 ends along the side (like a 4-noid), 1 end on the top, and 1 end on the bottom. So, M will look the same if it is rotated horizontally by $\frac{\pi}{2}$ and if it is rotated vertically by $\frac{\pi}{2}$. Determine G and dh for this surface.

Optional

Small Project 2.108. For Scherk's singly periodic surface the four ends are symmetrically placed so that if the surface is rotated by $\frac{\pi}{2}$ you will get the same image. This is because the denominator of dh is $z^4 - 1$ which has zeros that are equally spaced on the unit circle. It is possible to create a variation of Scherk's singly periodic surface that has four ends with rotational symmetry of π. That is, if the ends are labelled $E_1, \ldots, E_4$, then E_2 will be closer to E_1 than to E_3 (and likewise, E_4 will be closer to E_3 than to E_1) and if the surface is rotated by π you will get the same image. Determine G and dh for this surface.

Optional

Large Project 2.109. Describe and classify the possible minimal surfaces where one of the coordinates of the parametrization is fixed while the other two coordinates vary. For example, if x_3 is fixed to a specific function, what are the possible coordinate functions for x_1 and x_2? Try to generalize this approach as much as possible.

Optional

Large Project 2.110. Describe and classify the possible minimal surfaces with $G(z) = z^m$ and $dh = z^n\, dz$, for all $n, m \in \mathbb{N}$ (see Example 2.88 and Example 2.94). There are several distinct cases to consider. Determine how to separate m, n into these distinct cases remembering to discuss types of surface, types of ends, lines of symmetry, etc.

Optional

2.6. Minimal Surfaces and Harmonic Univalent Mappings

In the Summary on page 146 before the first Weierstrass representation, we learned that each coordinate function of the parametrization $\mathbf{x}$ of a minimal surface had the form $x_k = \operatorname{Re} \int \varphi_k dz$ with φ_k being analytic. Since the real part of an analytic function is a harmonic function, we see that each x_k is harmonic. Also from this Summary, we have that $(\varphi_1)^2 + (\varphi_2)^2 + (\varphi_3)^2 = 0$. This means that if we know the functions φ_1 and φ_2, we can determine the function φ_3. So, another way to get a Weierstrass representation for minimal surfaces is to use two harmonic functions x_1 and x_2. In other words, we can investigate minimal surfaces by studying harmonic mappings in the complex plane. Such mappings are known as planar harmonic mappings and have been studied independently of minimal surfaces.

In this section we will develop another Weierstrass representation. In this case we will use planar harmonic mappings instead of p and q as in Section 2.4 or G and dh as in Section 2.5. What benefit do we obtain from this new approach? First, we can use results about planar harmonic mappings to prove results about minimal surfaces. Since these two areas of mathematics developed independently, this approach can be fruitful. Second, in the study of planar harmonic mappings, much of the work deals with $1-1$ functions. That is, f will be $1-1$ in G means that if $f(z_1) = f(z_2)$, then $z_1 = z_2$. Geometrically, this means that the image, $f(G)$, will not overlap or intersect itself. When such functions are lifted from the complex plane into $\mathbb{R}^3$, the resulting surfaces are minimal graphs and hence have no self-intersections. This can be beneficial in establishing the embeddedness of the resulting minimal surfaces. For example, the $1-1$ planar harmonic function given by

$$f(z) = h(z) + \overline{g(z)} = \operatorname{Re}\left[\frac{i}{2}\log\left(\frac{i+z}{i-z}\right)\right] + i\operatorname{Im}\left[\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right]$$

maps the unit disk onto a square region. This square region is the projection (i.e., shadow) of Scherk's doubly periodic surface onto the plane. In other words, we can lift

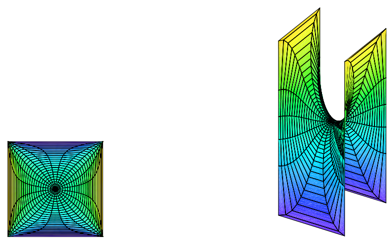


FIGURE 2.47. The image of $f(\mathbb{D})$ and Scherk's doubly periodic surface

f from $\mathbb{C}$ into $\mathbb{R}^3$ to get Scherk's doubly periodic surface. Planar harmonic mappings that are $1-1$ are also known as harmonic univalent mappings. Harmonic univalent mappings can be studied on their own without bringing in minimal surfaces and such a study is the topic of Chapter 4 of this book.

EXERCISE 2.111.

(a). Although all minimal graphs are embedded, the converse is not true. Give an example of an embedded minimal surface that is not a minimal graph.

(b). Suppose you have a nonunivalent harmonic mapping. Why could it not be the projection of a minimal graph?

Try it out!

Now that we have given an overview of this section, let's briefly discuss harmonic univalent mappings. A planar harmonic mapping is a function $f = u(x,y) + iv(x,y)$ where u and v are real harmonic functions. This concept is more general than that of an analytic function, because we do not require u and v to be harmonic conjugates. However, the following theorem allows us to relate a planar harmonic mapping to analytic functions. For our purposes, we will assume that the domain of f is the unit disk, $\mathbb{D}$.

THEOREM 2.112. Define a function $f = u + iv$, where u and v are real harmonic functions. If D is a simply-connected domain and $f : D \to \mathbb{C}$, then there exist analytic functions h and g such that $f = h + \overline{g}$.

EXERCISE 2.113.

(a) Show that $f(x,y) = u(x,y)+iv(x,y) = \left(\frac{1}{3}x^3 - xy^2 + x\right)+i\left(\frac{1}{3}y^3 - x^2y + y\right)$ is complex-valued harmonic by showing that u and v are real harmonic functions.
(b) Using $x = \frac{1}{2}(z+\overline{z})$ and $y = \frac{1}{2i}(z-\overline{z})$, rewrite $f(x,y) = \left(\frac{1}{3}x^3 - xy^2 + x\right) + i\left(\frac{1}{3}y^3 - x^2y + y\right)$ in terms of z and $\overline{z}$.
(c) Determine the analytic functions h and g such that $f = h + \overline{g}$.

Try it out!

Example 2.114. In the previous exercise, we saw that the planar harmonic map $f : \mathbb{D} \to \mathbb{C}$ defined by

$$f(x,y) = u(x,y) + iv(x,y) = \left(\frac{1}{3}x^3 - xy^2 + x\right) + i\left(\frac{1}{3}y^3 - x^2y + y\right)$$

can be written as

$$f(z) = h(z) + \overline{g}(z) = z + \frac{1}{3}\overline{z}^3.$$

What is the image of $\mathbb{D}$ under f? It is a hypocycloid with 4 cusps. This fact can be computed by considering $f(e^{i\theta}) = u(\theta)+iv(\theta)$ and comparing the component functions, $u(\theta)$ and $v(\theta)$, to the parametrized equation for a hypocycloid with 4 cusps. To help us visualize the image, we can use the applet *ComplexTool*. To graph the image of $\mathbb{D}$ under the harmonic function $f(z) = z + \frac{1}{3}\overline{z}^3$, enter this function in *ComplexTool* in the form $z + 1/3$ `conj` $(z \wedge 3)$ (see Figure 2.48). Remember this example; we will show that this function is related to a minimal graph.

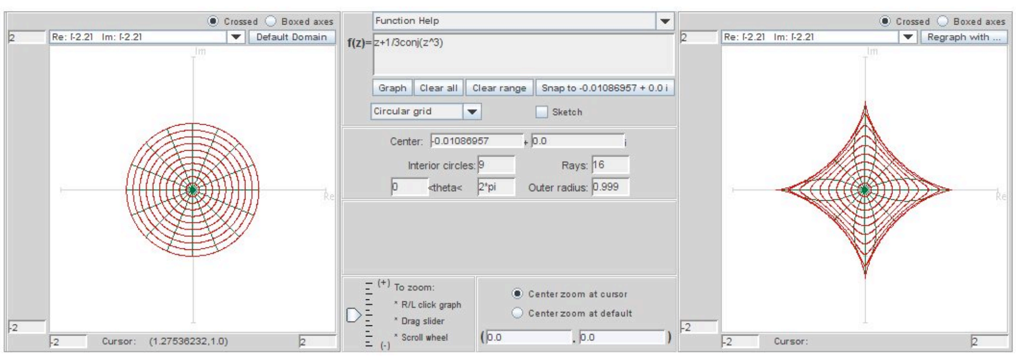


Figure 2.48. Image of $\mathbb{D}$ under the harmonic function $f(z) = z + \frac{1}{3}\overline{z}^3$

Note that the harmonic function $f(z) = h(z) + \overline{g}(z)$ can also be written in the form

(17) $$f(z) = \text{Re}\left\{h(z) + g(z)\right\} + i\,\text{Im}\left\{h(z) - g(z)\right\}.$$

This is because,

$$\text{Re}\{h+g\} = \frac{1}{2}[(h+g) + \overline{(h+g)}] \quad \text{and} \quad \text{Im}\{h-g\} = \frac{1}{2i}[(h-g) - \overline{(h-g)}].$$

Hence, in the previous example, $f(z) = z + \frac{1}{3}\overline{z}^3$ can also be written as $f(z) = \mathrm{Re}\left\{z + \frac{1}{3}z^3\right\} + i\,\mathrm{Im}\left\{z - \frac{1}{3}z^3\right\}$.

We are interested in harmonic functions that are $1-1$ or univalent, because this is one necessary condition in order to lift the harmonic mapping to a minimal graph. One theorem that establishes univalency requires the following background material.

Definition 2.115. The *dilatation* of $f = h + \overline{g}$ is $\omega(z) = g'(z)/h'(z)$.

Theorem 2.116 (Lewy [**11**]). $f = h + \overline{g}$ is locally univalent and orientation-preserving if and only if $|g'(z)/h'(z)| < 1$, for all $z \in \mathbb{D}$.

See Chapter 4 of this book for more background on the dilatation and on orientation-preserving.

Exercise 2.117. Show that if $z \in \mathbb{D}$, then $|\omega(z)| < 1$ for:

(a) $\omega_1(z) = e^{i\theta}z$, where $\theta \in \mathbb{R}$;

(b) $\omega_2(z) = z^n$, where $n = 1, 2, 3, \ldots$;

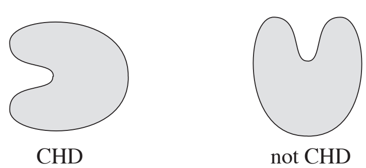
(c) $\omega_3(z) = \dfrac{z+a}{1+\overline{a}z}$, where $|a| < 1$;

(d) $\omega_4(z)$ being the composition of any of the functions ω above.

Try it out!

Creating nontrivial examples of harmonic univalent mappings that lift to minimal graphs is not easy. However, one way to do this is to use the shearing technique of Clunie and Sheil-Small. Before we proceed, we need to discuss a certain type of domain.

Definition 2.118. A domain Ω is *convex in the direction of the real axis* (or convex in the horizontal direction, CHD) if every line parallel to the real axis has a connected intersection with Ω.



Theorem 2.119 (Clunie and Sheil-Small). A harmonic function $f = h + \overline{g}$ locally univalent in $\mathbb{D}$ is a univalent mapping of $\mathbb{D}$ onto a CHD domain $\Longleftrightarrow$ $h - g$ is an analytic univalent mapping of $\mathbb{D}$ onto a CHD domain.

Remark 2.120. This technique is known as the "shear" method or "shearing" a function. In our situation, suppose $F = h - g$ is an analytic univalent function convex in the real direction. Then the corresponding harmonic shear is

$$f = h + \overline{g} = h - g + g + \overline{g} = h - g + 2\,\mathrm{Re}\{g\}.$$

So, the harmonic shear differs from the analytic function by adding a real function to it. Geometrically, you can think of this as taking F, the original analytic univalent function convex in the real direction, and cutting it up into thin horizontal slices which are then translated and/or scaled in a continuous way to form the corresponding harmonic function, f. This is why the method is called "shearing." Since F is univalent and convex in the real direction and we are only adding a continuous real function to it, the univalency is preserved.

Example 2.121. Let

$$h(z) - g(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) \tag{18}$$

which is an analytic function that maps $\mathbb{D}$ onto a horizontal strip convex in the direction of the real axis (see Figure 2.49).

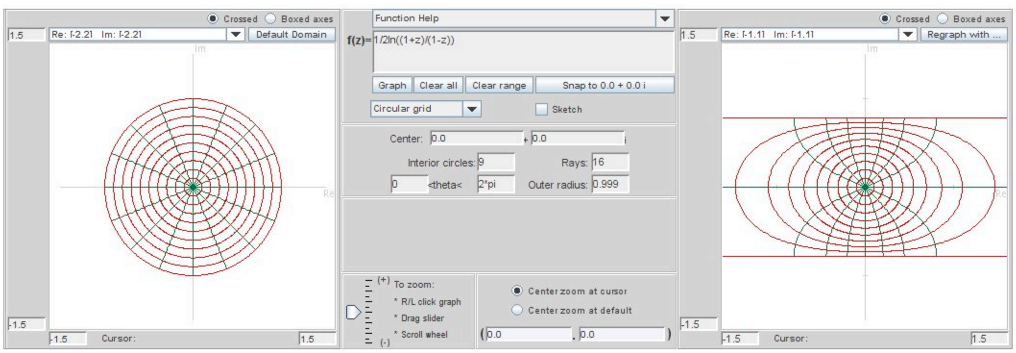


Figure 2.49. Image of $\mathbb{D}$ under the analytic function $\dfrac{1}{2}\log\left(\dfrac{1+z}{1-z}\right)$

Let

$$\omega(z) = g'(z)/h'(z) = -z^2.$$

Applying the shearing method from Theorem 2.119 with the substitution $g'(z) = -z^2h'(z)$, we have

$$h'(z) - g'(z) = \frac{1}{1-z^2} \Rightarrow h'(z) + z^2h'(z) = \frac{1}{1-z^2}$$
$$\Rightarrow h'(z) = \frac{1}{1-z^4} = \frac{1}{4}\left[\frac{1}{1+z} + \frac{1}{1-z} + \frac{i}{i+z} + \frac{i}{i-z}\right].$$

Integrating $h'(z)$ and normalizing so that $h(0) = 0$, yields

$$h(z) = \frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \frac{i}{4}\log\left(\frac{i+z}{i-z}\right). \tag{19}$$

We can use this same method to solve for normalized $g(z)$, where $g(0) = 0$. Note that we can also find $g(z)$ by using eqs. (56) and (57). Either way, we get

$$g(z) = -\frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \frac{i}{4}\log\left(\frac{i+z}{i-z}\right).$$

So

$$f(z) = h(z) + \overline{g(z)} = \operatorname{Re}\left[\frac{i}{2}\log\left(\frac{i+z}{i-z}\right)\right] + i\operatorname{Im}\left[\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right].$$

What is $f(\mathbb{D})$? Notice that

$$f(z) = \left[-\frac{1}{2}\arg\left(\frac{i+z}{i-z}\right)\right] + i\left[\frac{1}{2}\arg\left(\frac{1+z}{1-z}\right)\right] = u + iv.$$

Let $z = e^{i\theta} \in \partial\mathbb{D}$. Then

$$\frac{i+z}{i-z} = \frac{i+e^{i\theta}}{i-e^{i\theta}}\frac{-i-e^{-i\theta}}{-i-e^{-i\theta}} = \frac{1-i(e^{i\theta}+e^{-i\theta})-1}{1+i(e^{i\theta}-e^{-i\theta})+1} = -i\frac{\cos\theta}{1-\sin\theta}.$$

Thus,

$$u = -\frac{1}{2}\arg\left(\frac{i+z}{i-z}\right)\bigg|_{z=e^{i\theta}} = \begin{cases} \frac{\pi}{4} & \text{if } \cos\theta > 0, \\ -\frac{\pi}{4} & \text{if } \cos\theta < 0. \end{cases}$$

Likewise, we can show that

$$v = \begin{cases} \frac{\pi}{4} & \text{if } \sin\theta > 0, \\ -\frac{\pi}{4} & \text{if } \sin\theta < 0. \end{cases}$$

Therefore, we have that $z = e^{i\theta} \in \partial\mathbb{D}$ is mapped to

$$u + iv = \begin{cases} z_1 = \frac{\pi}{2\sqrt{2}}\, e^{i\frac{\pi}{4}} = \frac{\pi}{4} + i\frac{\pi}{4} & \text{if } \theta \in (0, \frac{\pi}{2}), \\ z_2 = \frac{\pi}{2\sqrt{2}}\, e^{i\frac{3\pi}{4}} = -\frac{\pi}{4} + i\frac{\pi}{4} & \text{if } \theta \in (\frac{\pi}{2}, \pi), \\ z_3 = \frac{\pi}{2\sqrt{2}}\, e^{i\frac{5\pi}{4}} = -\frac{\pi}{4} - i\frac{\pi}{4} & \text{if } \theta \in (\pi, \frac{3\pi}{2}), \\ z_4 = \frac{\pi}{2\sqrt{2}}\, e^{i\frac{7\pi}{4}} = \frac{\pi}{4} - i\frac{\pi}{4} & \text{if } \theta \in (\frac{3\pi}{2}, 2\pi). \end{cases}$$

Thus, this harmonic function maps $\mathbb{D}$ onto the interior of the region bounded by a square with vertices at z_1, z_2, z_3 and z_4.

EXERCISE 2.122. Verify that shearing $h(z) - g(z) = z - \frac{1}{3}z^3$ with $\omega(z) = z^2$ yields $f(z) = z + \frac{1}{3}\overline{z}^3$ from Example 2.114.

Try it out!

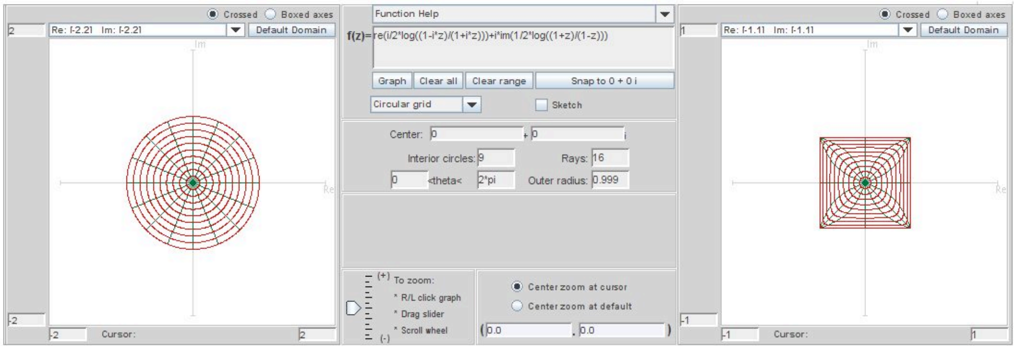


FIGURE 2.50. Image of $\mathbb{D}$ under $f(z) = \operatorname{Re}\left[\frac{i}{2}\log\left(\frac{i+z}{i-z}\right)\right] + i\operatorname{Im}\left[\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right]$.

To actually find the minimal graph that is associated with specific types of harmonic univalent mappings, we need to develop the appropriate Weierstrass representation as outlined in eq (13). Recall that it must satisfy the properties $\phi^2 = 0$ and $|\phi|^2 \neq 0$, and we want it to use planar harmonic mappings. A natural choice is to consider

$$\begin{aligned} x_1 &= \operatorname{Re}(h+g) = \operatorname{Re}\int (h'+g')\,dz = \operatorname{Re}\int \varphi_1\,dz \\ x_2 &= \operatorname{Im}(h-g) = \operatorname{Re}\int -i(h'-g')\,dz = \operatorname{Re}\int \varphi_2\,dz \\ x_3 &= \operatorname{Re}\int \varphi_3\,dz \end{aligned}$$

and then solve for φ_3.

EXERCISE 2.123. Derive that $\varphi_3 = 2ih'\sqrt{g'/h'} = 2i\sqrt{g'h'}$.
Try it out!

We need φ_3 to be analytic and so we require the dilatation $\omega = g'/h'$ to be a perfect square.

THEOREM 2.124 (Weierstrass Representation (h,g)). If $f = h + \overline{g}$ is a sense-preserving harmonic univalent mapping of $\mathbb{D}$ onto some domain $\Omega \subset \mathbb{C}$ with dilatation $\omega = q^2$ for some function q analytic in $\mathbb{D}$, then the isothermal parametrization

$$\begin{aligned} \mathbf{x}(u,v) =& \big(x_1, x_2, x_3\big) \\ =& \left(\operatorname{Re}\{h(z)+g(z)\}, \operatorname{Im}\{h(z)-g(z)\}, 2\operatorname{Im}\left\{\int_0^z \sqrt{g'(\zeta)h'(\zeta)}d\zeta\right\}\right) \end{aligned}$$

defines a minimal graph whose projection onto the complex plane is f. Conversely, if a minimal graph $\mathbf{x}(u,v) = \big\{(u,v,F(u,v)) : u+iv \in \Omega\big\}$ is parametrized by sense-preserving isothermal parameters $z = x+iy \in \mathbb{D}$, then the projection onto its base plane defines a harmonic univalent mapping $f(z) = u + iv = \operatorname{Re}\{h(z) + g(z)\} + i\operatorname{Im}\{h(z) - g(z)\}$ of $\mathbb{D}$ onto Ω whose dilatation is the square of an analytic function.

Summary: Let $f = h + \overline{g}$ defined on $\mathbb{D}$ be a harmonic univalent mapping such that the dilatation $\omega = g'/h'$ is the square of an analytic function and $|\omega(z)| < 1$ for all $z \in \mathbb{D}$. Then f lifts to a minimal graph using the Weierstrass formula given in Theorem 2.124.

Example 2.125. Recall from Example 2.114 the harmonic univalent mapping

$$f(z) = z + \frac{1}{3}\overline{z}^3 = \operatorname{Re}\left(z + \frac{1}{3}z^3\right) + i\operatorname{Im}\left(z - \frac{1}{3}z^3\right).$$

Note that $h(z) = z$ and $g(z) = \frac{1}{3}z^3$. Also, $\omega(z) = z^2$ which is the square of an analytic function. Hence this harmonic mapping lifts to a minimal graph. We compute that

$$x_3 = 2\operatorname{Im}\int_0^z \sqrt{h'(\zeta)g'(\zeta)}\,d\zeta = \operatorname{Im}\left(z^2\right).$$

This yields a parametrization of a surface that is the conjugate of the Enneper's surface given in Example 2.67:

$$\mathbf{x} = \left(\operatorname{Re}\left\{z + \frac{1}{3}z^3\right\}, \operatorname{Im}\left\{z - \frac{1}{3}z^3\right\}, \operatorname{Im}\left\{z^2\right\}\right)$$

and hence yields Enneper's surface. Note that the projection of the Enneper surface onto the x_1x_2-plane is the image of $\mathbb{D}$ under the harmonic mapping f. Also, while Enneper's surface is not a graph over $\mathbb{C}$, it is a graph over $\mathbb{D}$ as this result proves. You can see this by using *MinSurfTool* with the **W.E. (h,g)** tab. Enter in the functions $h(z) = z$, $g(z) = \frac{1}{3}z^3$, and $\sqrt{h' \cdot g'} = z$. Make sure the `Surface and projection` button is clicked (this will allow you to see both the minimal surface and its projection that is related to the harmonic mapping). Also, use the Disk domain for the unit disk (i.e., radius min: 0; radius max: 1; theta min: 0; theta max: 2 pi). The minimal surface is colored purple while the $f(\mathbb{D})$ is colored green (see Figure 2.51). As you move the image so that it is viewed from the top, the projection of the minimal surface matches the image of $f(\mathbb{D})$ (see Figure 2.52).

Exploration 2.126. In the Weierstrass representation (h,g), we require that $\omega = g'/h'$ be the square of an analytic function. This is necessary because $\varphi_3 = ih'\sqrt{g'/h'}$, and if g'/h' were not the square of an analytic function, then there would be two branches of the square root. Geometrically, we can see that this is necessary. Use *MinSurfTool* with the **W.E. (h,g)** tab to graph the following images and describe why the geometry of those functions $f = h + \overline{g}$ in the left column do lift to a minimal graph

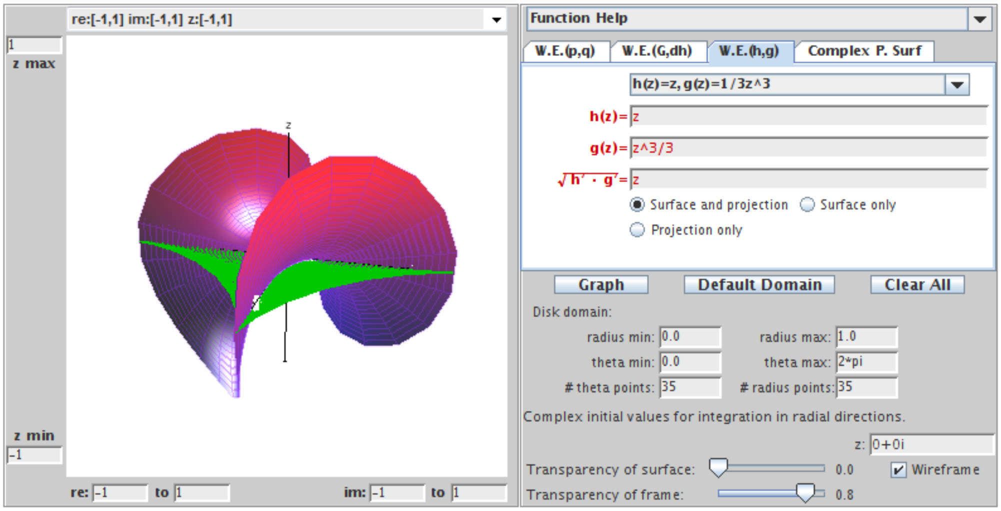


FIGURE 2.51. Side view of the Enneper surface and the image of the unit disk under the harmonic map.

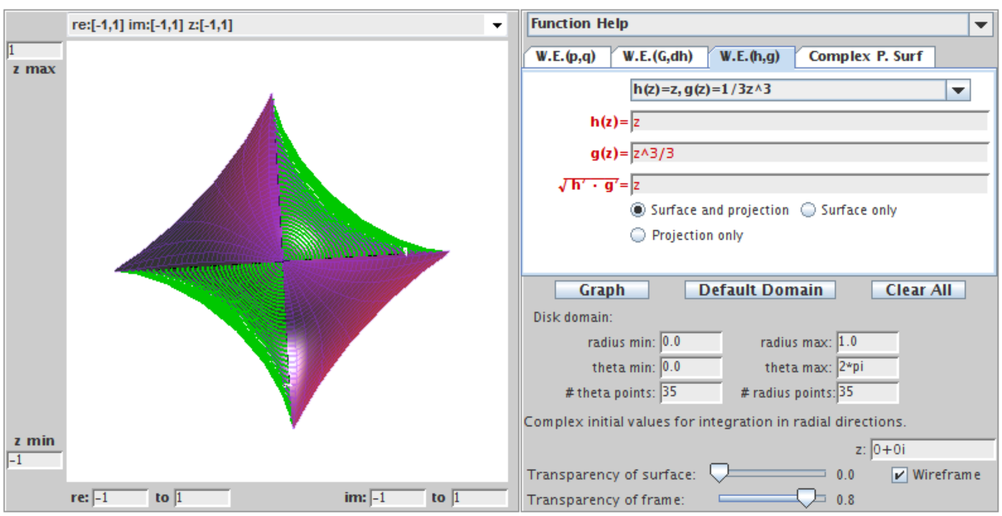


FIGURE 2.52. The projection of the Enneper surface is the image of the unit disk under the harmonic map.

while those in the right column do not. To enter $\sqrt{z}$ into the applet type `sqrt(z)`.

(a) $z + \frac{1}{3}\overline{z}^3$;
(note: $\omega = z^2$ and $\sqrt{h' \cdot g'} = z$)

(b) $z + \frac{1}{2}\overline{z}^2$;
(note: $\omega = z$ and $\sqrt{h' \cdot g'} = \sqrt{z}$)

(c) $z - \frac{1}{5}\overline{z}^5$;
(note: $\omega = -z^4$ and $\sqrt{h' \cdot g'} = iz^2$)

(d) $z - \frac{1}{4}\overline{z}^4$.
(note: $\omega = -z^5$ and $\sqrt{h' \cdot g'} = i\sqrt{z^5}$)

Example 2.127. Consider the harmonic univalent mapping from Example 2.121 given by

$$f(z) = h(z) + \overline{g(z)} = \operatorname{Re}\left[\frac{i}{2}\log\left(\frac{i+z}{i-z}\right)\right] + i\operatorname{Im}\left[\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right].$$

Because $\omega(z) = -z^2$ is the square of an analytic function, we can lift this harmonic mapping to a minimal graph. We compute that

$$x_3 = 2\operatorname{Im}\int_0^z \frac{iz}{1-z^4}\, d\zeta = \frac{1}{2}\operatorname{Im}\left\{i\log\left(\frac{1+z^2}{1-z^2}\right)\right\}.$$

This yields a parametrization of Scherk's doubly periodic minimal surface:

$$\mathbf{x} = \left(\operatorname{Re}\left[\frac{i}{2}\log\left(\frac{i+z}{i-z}\right)\right], \operatorname{Im}\left[\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right], \frac{1}{2}\operatorname{Im}\left[i\log\left(\frac{1+z^2}{1-z^2}\right)\right]\right).$$

We can use **Pre-set functions** from the **W.E. (h,g)** tab on *MinSurfTool* to plot the minimal graph and the image of the unit disk under the planar harmonic mapping, where

$$\begin{aligned} h(z) =& \frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \frac{i}{4}\log\left(\frac{i+z}{i-z}\right), \\ g(z) =& -\frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \frac{i}{4}\log\left(\frac{i+z}{i-z}\right). \end{aligned}$$

Because of the singularities at $\pm 1, \pm i$, the radius max is set to 0.999. Notice that the projection of Scherk's doubly periodic surface onto the x_1x_2-plane is a square which is the image of $\mathbb{D}$ under the harmonic mapping f.

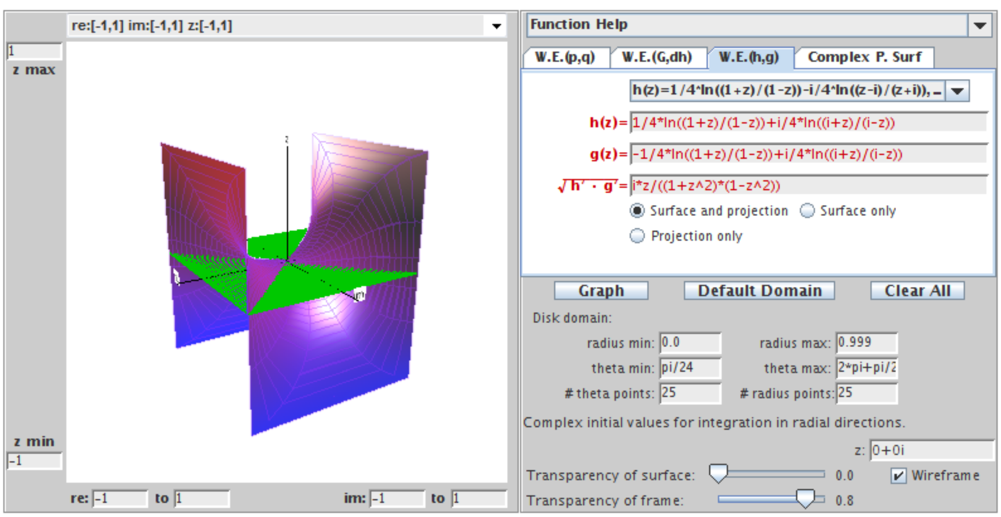


Figure 2.53. Side view of Scherk's doubly periodic surface and the image of the unit disk under the harmonic map.

Exploration 2.128. Use the **Pre-set functions** from the **W.E. (h,g)** tab on *MinSurfTool* to plot the minimal graphs associated with the given functions h and g for the planar harmonic mappings. Determine which minimal surfaces these are.

(a) $h(z) = z$, $g(z) = \frac{1}{2n+1} z^{2n+1} (n = 1, 2, 3, \ldots)$;

(b) $h(z) = z$, $g(z) = \frac{1}{z}$;

(c) $h(z) = z$, $g(z) = -\frac{1}{z}$;

(d) $h(z) = \frac{1}{5} z^5$, $g(z) = -\frac{1}{z}$;

(e) $h(z) = \frac{1}{4} \log\left(\frac{i+z}{i-z}\right) - \frac{i}{4} \log\left(\frac{1+z}{1-z}\right)$, $g(z) = \frac{1}{4} \log\left(\frac{i+z}{i-z}\right) + \frac{i}{4} \log\left(\frac{1+z}{1-z}\right)$.

Try it out!

If we start with a harmonic univalent mapping that has a perfect square dilatation we can use Theorem 2.124 to find the parametrization of the corresponding minimal graph. In Example 2.125, we saw that the minimal graph corresponding to the harmonic univalent map $f(z) = z + \frac{1}{3}\overline{z}^3$ is Enneper's surface, and in Example 2.127 the minimal graph corresponding to the harmonic univalent map $f(z) = \mathrm{Re}[\frac{i}{2} \log(\frac{i+z}{i-z})] + i\,\mathrm{Im}[\frac{1}{2} \log(\frac{1+z}{1-z})]$ is Scherk's doubly periodic surface. There have been several research papers in the field of harmonic univalent mappings that have used Theorem 2.124 to create minimal graphs from harmonic univalent mappings (e.g., [**8**], [**9**], [**10**], [**11**], [**17**]). However, many of them have not identified the specific minimal graph created, because it is often not easy to do so as it was in Examples 2.125 and 2.127.

Question: Given a harmonic univalent mapping we can use Theorem 2.124 to find the parametrization of a minimal graph. Can we determine which minimal graph this is?

The following examples will show how can determine the minimal graph when the parametrization is not a standard parametrization for a known minimal surface.

Example 2.129. The analytic function,

$$F(z) = \frac{z}{1-z},$$

is an important and interesting function in complex analysis. It maps $\mathbb{D}$ onto the right half-plane $\{z \in \mathbb{C} \big| \mathrm{Re}\{z\} > 0\}$. If we shear

$$h(z) - g(z) = \frac{z}{1-z} \text{ with } \omega(z) = g'(z)/h'(z) = z^2,$$

then we get the harmonic univalent mapping $f = h + \overline{g}$, where

$$h = \frac{1}{8} \log\left(\frac{z+1}{z-1}\right) + \frac{3z - 2z^2}{4(1-z)^2} \text{ and } g = \frac{1}{8} \log\left(\frac{z+1}{z-1}\right) - \frac{z - 2z^2}{4(1-z)^2}.$$

The image of $\mathbb{D}$ under $f = h + \overline{g}$ is shown in Figure 2.54.

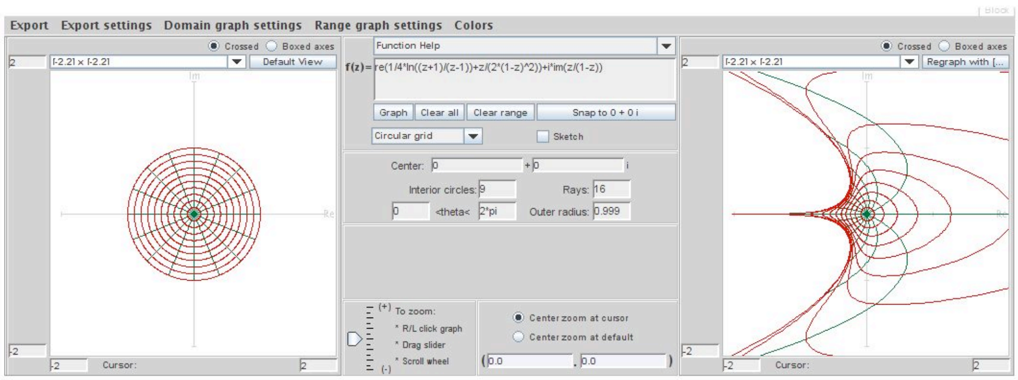


FIGURE 2.54. Image of $\mathbb{D}$ under $f = \operatorname{Re}\left(\frac{1}{4}\log\left(\frac{z+1}{z-1}\right) + \frac{z}{2(1-z)^2}\right) + \operatorname{Im}\left(\frac{z}{1-z}\right)$.

Because $\omega(z) = z^2$, we can use Theorem 2.124 to find the parametrization of the corresponding minimal graph:

$$\mathbf{x} = \left(\operatorname{Re}\left\{\frac{1}{4}\log\left(\frac{z+1}{z-1}\right)+\frac{z}{2(1-z)^2}\right\}, \operatorname{Im}\left\{\frac{z}{1-z}\right\}, \operatorname{Im}\left\{\frac{1}{4}\log\left(\frac{z+1}{z-1}\right)-\frac{z}{2(1-z)^2}\right\}\right).$$

What minimal graph is this? It is difficult to tell. The coordinate functions do not look like a standard parametrization for a known minimal surface. Let us try to use a substitution to rewrite this parametrization into a form that allows us to identify the minimal graph. If we use a Möbius transformation for the substitution, it will not affect the geometry of the minimal graph. Letting $z \mapsto \frac{\hat{z}+1}{\hat{z}-1}$, we get the parametrization:

$$\hat{\mathbf{x}} = \left(\frac{1}{4}\operatorname{Re}\left\{\log(\hat{z}) + \frac{1}{2}\hat{z}^2 - \frac{1}{2}\right\}, -\frac{1}{2}\operatorname{Im}\{\hat{z}\}, -\frac{1}{4}\operatorname{Im}\left\{\log(\hat{z}) - \frac{1}{2}\hat{z}^2 + \frac{1}{2}\right\}\right).$$

This is useful, because it simplifies the log terms in x_1 and x_3. Also, notice that by switching the coordinate functions, and factoring out $\frac{1}{2}$ we have:

$$\text{(20)} \quad \widetilde{\mathbf{x}} = \left(-\frac{1}{2}\left[\frac{1}{2}\operatorname{Im}\left\{\log(\widetilde{z}) - \frac{1}{2}\widetilde{z}^2\right\}\right], \frac{1}{2}\left[\frac{1}{2}\operatorname{Re}\left\{\log(\widetilde{z}) + \frac{1}{2}\widetilde{z}^2\right\}\right], -\frac{1}{2}\left[\operatorname{Im}\{\widetilde{z}\}\right]\right),$$

and this looks close to the standard parametrization for the wavy plane. In Figure 2.55 we have used the **Complex P. Surf** tab in *MinSurfTool* to graph the image of the surface with this parametrization.

In fact, the coordinate functions above correspond to the conjugate surface of the wavy plane scaled by $\frac{1}{2}$. This is clear given the actual coordinates of the wavy plane

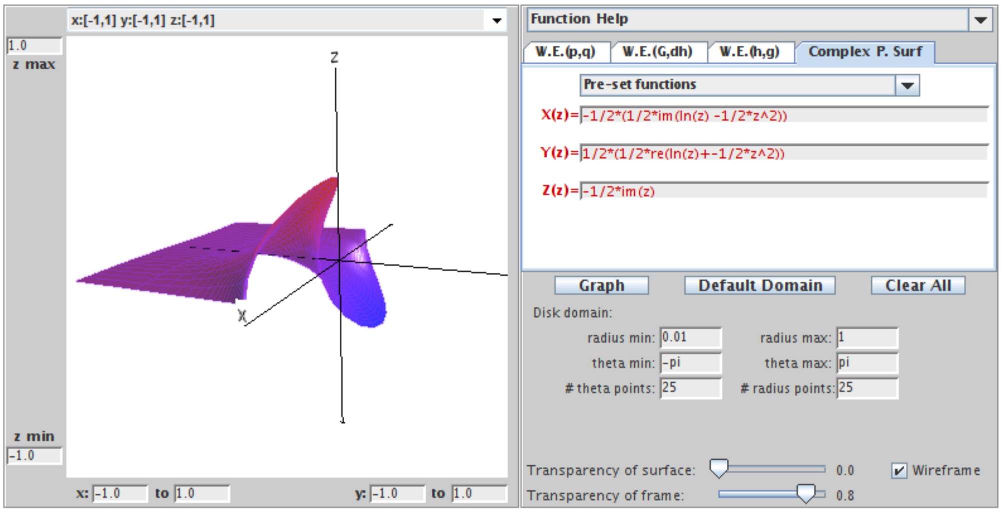


FIGURE 2.55. The conjugate of the wavy plane.

below:

$$\mathbf{W} = \left(\frac{1}{2}\operatorname{Re}\left\{\log(z) - \frac{1}{2}z^2\right\}, -\frac{1}{2}\operatorname{Im}\left\{\log(z) + \frac{1}{2}z^2\right\}, \operatorname{Re}\{z\}\right).$$

Since the wavy plane is its own conjugate surface, this means that it is accurate to describe our original minimal graph as the wavy plane.

How does this match with the image in Figure 2.54 of $\mathbb{D}$ under the original harmonic univalent map? In deriving the familiar parametrization of the wavy plane, we applied the transformation $z \mapsto \frac{\hat{z}+1}{\hat{z}-1}$, switched the coordinate functions, and took the conjugate function. These changes are equivalent to altering the original domain $\mathbb{D}$, so we are looking at a piece of the wavy plane coming from a different domain G. The projection of the wavy plane coming from G is shown in Figure 2.56. Notice the similarity between the image in Figure 2.54 and the image in Figure 2.56.

EXERCISE 2.130. An important function in complex analysis is the Koebe function given by $\frac{z}{(1-z)^2}$. By shearing

$$h(z) - g(z) = \frac{z}{(1-z)^2} \quad \text{with} \quad \omega(z) = z^2,$$

we derive the harmonic univalent mapping $f = h + \overline{g}$, where

$$h = \frac{z - z^2 + \frac{1}{3}z^3}{(1-z)^3} \quad \text{and} \quad g = \frac{\frac{1}{3}z^3}{(1-z)^3}.$$

(a) Use Theorem 2.124 to find the parametrization of the minimal graph that f lifts to.

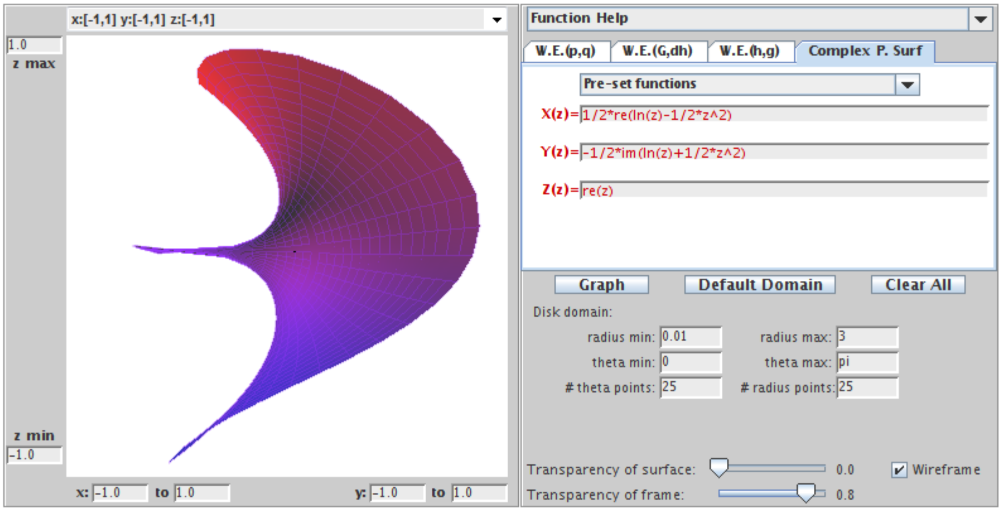


FIGURE 2.56. The wavy plane mapped from domain G.

(b) Use the approach of Example 2.129 to show analytically that this minimal graph is Enneper's surface.
(c) Use *ComplexTool* to graph the image of $\mathbb{D}$ under f and *MinSurfTool* to sketch the corresponding minimal graph.

Try it out!

EXAMPLE 2.131. By shearing $h(z) - g(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ with $\omega(z) = g'(z)/h'(z) = m^2z^2$, where $|m| \leq 1$, it was shown in [**9**] that the harmonic function $f = h + \overline{g}$ is univalent, where

$$h(z) = \frac{1}{2(1-m^2)}\log\left(\frac{1+z}{1-z}\right) + \frac{m}{2(m^2-1)}\log\left(\frac{1+mz}{1-mz}\right)$$
$$g(z) = \frac{m^2}{2(1-m^2)}\log\left(\frac{1+z}{1-z}\right) + \frac{m}{2(m^2-1)}\log\left(\frac{1+mz}{1-mz}\right).$$

When $m = e^{i\frac{\pi}{2}}$, the function f is the same as in Example 2.121 and the image of $\mathbb{D}$ under $f = h + \overline{g}$ is a square.

In fact, for every m such that $|m| = 1$, the image of $\mathbb{D}$ under $f = h + \overline{g}$ is a parallelogram.

Since $\omega(z) = g'(z)/h'(z) = m^2z^2$, we can lift f to a minimal graph. We compute that

$$x_3 = \operatorname{Im}\left\{\frac{m}{1-m^2}\log\left(\frac{1-m^2z^2}{1-z^2}\right)\right\}.$$

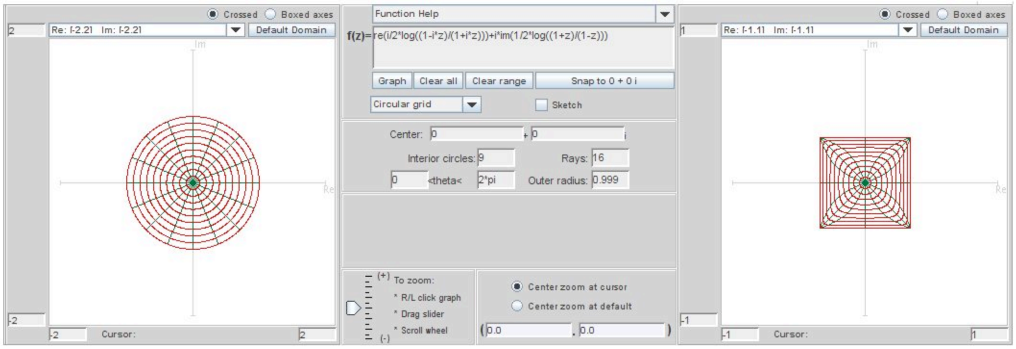


FIGURE 2.57. Image of $\mathbb{D}$ under $f = h + \overline{g}$ when $m = e^{i\frac{\pi}{2}}$.

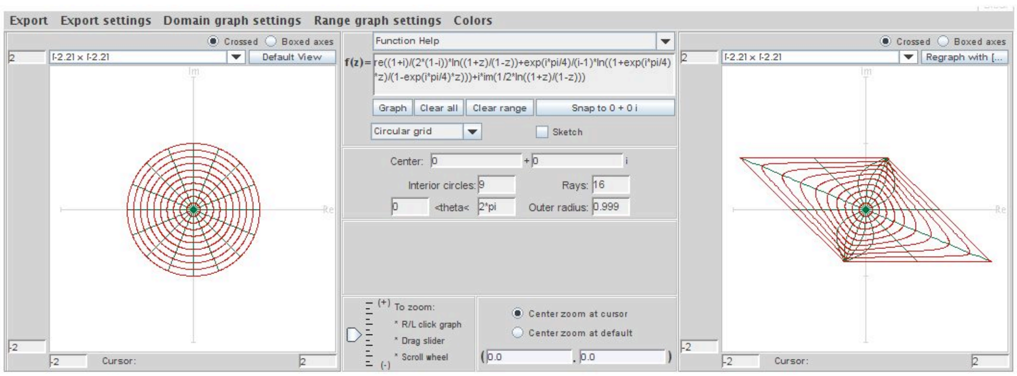


FIGURE 2.58. Image of $\mathbb{D}$ under $f = h + \overline{g}$ when $m = e^{i\frac{\pi}{4}}$.

Hence, the corresponding parametrization of the minimal graph is

$$\mathbf{x} = \Bigg(\operatorname{Re}\left\{ \frac{1+m^2}{2(1-m^2)} \log\left(\frac{1+z}{1-z}\right) + \frac{m}{(m^2-1)} \log\left(\frac{1+mz}{1-mz}\right) \right\},$$
$$\operatorname{Im}\left\{ \frac{1}{2} \log\left(\frac{1+z}{1-z}\right) \right\}, \operatorname{Im}\left\{ \frac{m}{1-m^2} \log\left(\frac{1-m^2z^2}{1-z^2}\right) \right\} \Bigg).$$

When $m = e^{i\frac{\pi}{2}}$, the minimal graph is Scherk's doubly periodic surface (see Figure 2.59).

For $m = e^{i\theta}, (0 < \theta < \frac{\pi}{2})$, the minimal graphs are slanted Scherk's surfaces (see Figure 2.60).

What is the minimal graph when $m = 1$? In the limit (i.e., $\theta = 0$) we have the equation

$$\mathbf{x} = \left(\operatorname{Re}\left\{ \frac{z}{1-z^2} \right\}, \operatorname{Re}\left\{ -\frac{i}{2} \log\left(\frac{1+z}{1-z}\right) \right\}, \operatorname{Re}\left\{ \frac{-iz^2}{1-z^2} \right\} \right).$$

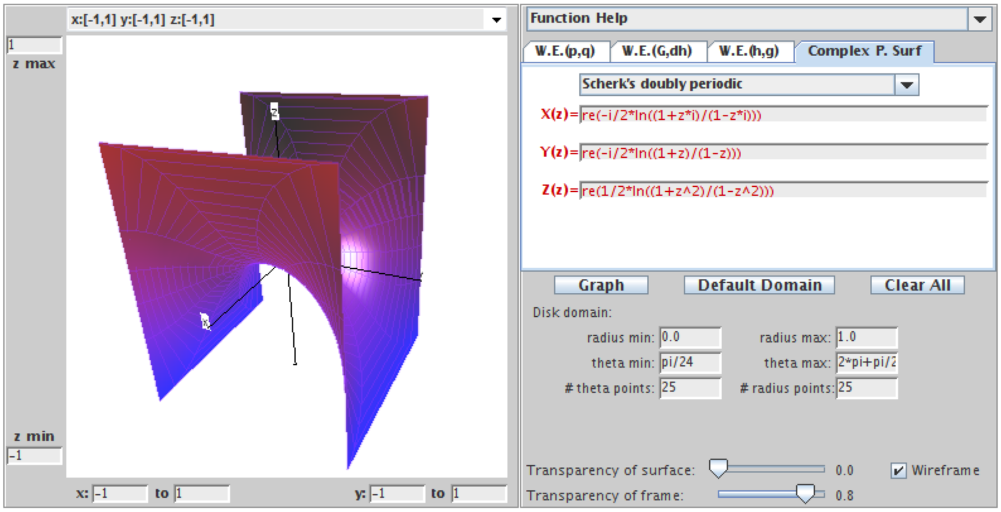


FIGURE 2.59. Scherk's doubly periodic square surface ($m = e^{i\frac{\pi}{2}}$).

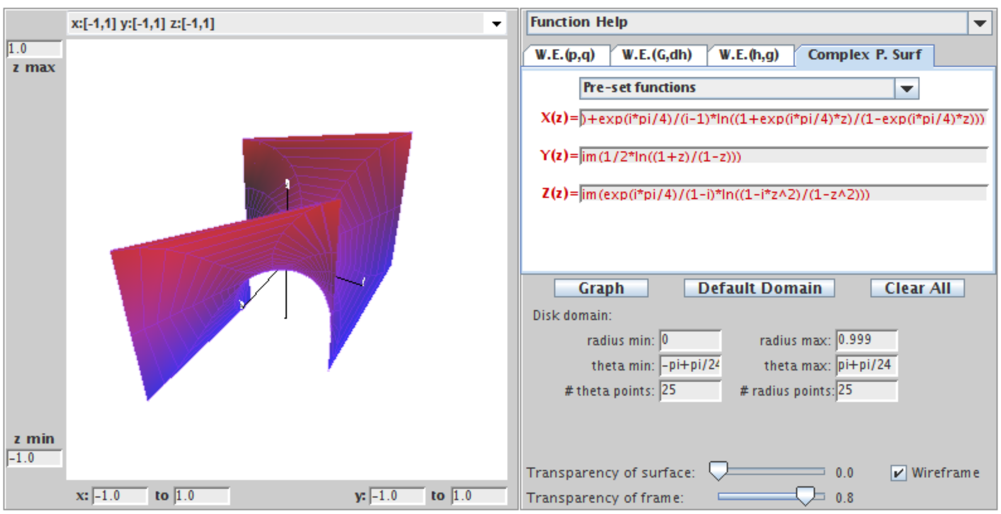


FIGURE 2.60. A Scherk's doubly periodic parallelogram surface ($m = e^{i\frac{\pi}{4}}$).

Again, this parametrization does not look like a standard parametrization of a known minimal surface. However, we can use a substitution to rewrite this parametrization into a form that allows us to identify the minimal graph. Using the substitution $z \longmapsto \frac{e^z-1}{e^z+1}$ and the fact that $\operatorname{Re}\left\{\frac{-iz^2}{1-z^2}\right\} = \operatorname{Re}\left\{\frac{1}{2i}\frac{1+z^2}{1-z^2}\right\}$, this equation is equivalent to

$$X = \left(\frac{1}{2}\sinh u \cos v, \frac{1}{2}v, \frac{1}{2}\sinh u \sin v\right),$$

which is an equation of a helicoid. Thus, we have a family of minimal surfaces that transform Scherk's doubly periodic surface continuously into the helicoid. This is neat!

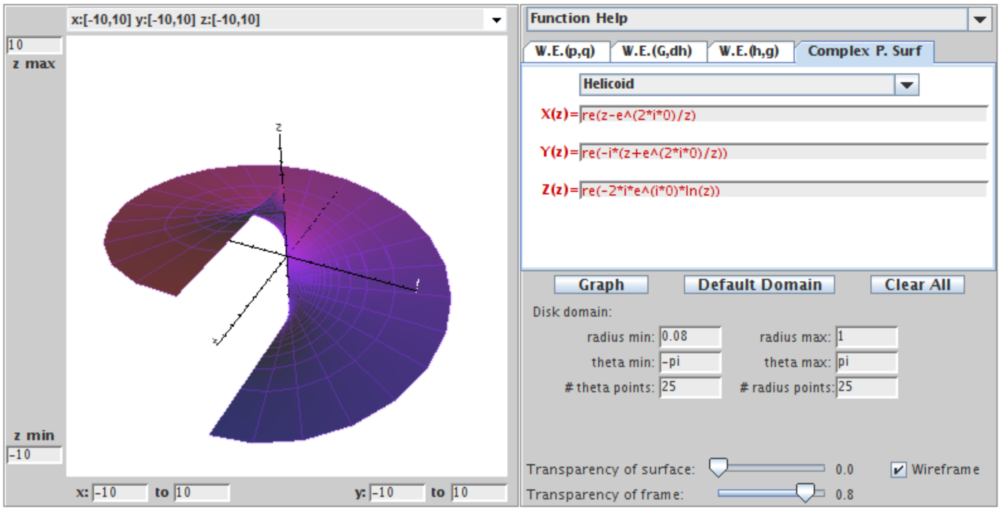


FIGURE 2.61. Side view of helicoid that is the limit function of the slanted Scherk's surfaces.

EXERCISE 2.132. Show that $\mathrm{Re}\left\{\dfrac{-iz^2}{1-z^2}\right\} = \mathrm{Re}\left\{\dfrac{1}{2i}\dfrac{1+z^2}{1-z^2}\right\}$.

Try it out!

EXERCISE 2.133. Consider the harmonic univalent map $f(z) = h(z) + \overline{g}(z)$, where

$$h = \frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \frac{\frac{1}{2}z}{1-z^2} \quad \text{and} \quad g = \frac{1}{4}\log\left(\frac{1+z}{1-z}\right) - \frac{\frac{1}{2}z}{1-z^2}.$$

(a) Use Theorem 2.124 to find the parametrization of the minimal graph that f lifts to.
(b) Use *ComplexTool* to graph the image of $\mathbb{D}$ under f and *MinSurfTool* with tab **W.E.(h,g)** to sketch the corresponding minimal graph.
(c) Use the approach of Example 2.131 to show analytically that this minimal graph is the catenoid.

Try it out!

LARGE PROJECT 2.134. The analytic function, $F(z) = z$, maps the unit disk, $\mathbb{D}$, onto itself. Shear $h(z) - g(z) = z$ with various perfect-square dilatations, ω, that satisfy the condition $|\omega| < 1$ for all $z \in \mathbb{D}$ (e.g., $\omega = z^{2n}(n \in \mathbb{N}), \omega = e^{i\theta}z^2(\theta \in \mathbb{R}), \omega = \left(\frac{z-a}{1-\bar{a}z}\right)^2(|a| < 1)$). Determine the corresponding minimal graphs.

Optional

Large Project 2.135. The analytic function, $F(z) = \frac{z}{1-z}$, maps the unit disk, $\mathbb{D}$, onto a right half-plane and is an important function. Shear $h(z) + g(z) = \frac{z}{1-z}$ with various perfect-square dilatations, ω, that satisfy the condition $|\omega| < 1$ for all $z \in \mathbb{D}$ (e.g., $\omega = z^{2n}(n \in \mathbb{N}), \omega = e^{i\theta}z^2(\theta \in \mathbb{R}), \omega = \left(\frac{z-a}{1-\bar{a}z}\right)^2(|a| < 1)$). Determine the corresponding minimal graphs.

Optional

2.7. Conclusion

We have presented an introduction to minimal surfaces and described a few topics that students can explore using the exercises, the exploratory problems, and the projects along with the applets. For a deeper and thorough explanation of differential geometry consult [**7**], [**18**], or [**21**] for beginners, and [**3**] for intermediates. Also, you should consider Spivak's five volume work [**24**]. Oprea's book [**22**] is a very nice source for an introduction to minimal surfaces. For more background on minimal surfaces we recommend [**25**], [**14**], [**15**], [**6**], [**23**], and [**20**].

2.8. Additional Exercises

Differential Geometry

EXPLORATION 2.136. An oblique cylinder can be parametrized by

$$\mathbf{x}(u,v) = (\cos u, \sin u + v\cos\theta, v\sin\theta),$$

where $\theta \in \left(0, \frac{\pi}{2}\right)$ is a fixed value. Use *DiffGeomTool* to explore what happens to the oblique cylinder as θ varies between 0 and $\frac{\pi}{2}$.

EXERCISE 2.137. Use *DiffGeomTool* to graph the surface parametrized by

$$\mathbf{x}(u,v) = \left(\cos u\left(1 + v\sin\left(\frac{1}{2}u\right)\right), \sin u\left(1 + v\sin\left(\frac{1}{2}u\right)\right), v\cos\left(\frac{1}{2}u\right)\right),$$

where $-\pi < u < \pi$, $-\frac{1}{2} < v < \frac{1}{2}$. This surface is known as the Möbius strip and is nonorientable; that is, the normal vector can change from pointing outward to pointing inward as it travels along a closed path on the surface. You can see this in *DiffGeomTool* by clicking on the **Normal vector** box and setting the `Point locator: (u,v)=` to $(\pi - 0.1, 0)$. Next, change this u coordinate to each of the following values: $u = \pi - 0.1 - 1$, $u = \pi - 0.1 - 2$, $u = \pi - 0.1 - 3$, $u = \pi - 0.1 - 4$, $u = \pi - 0.1 - 5$, and $u = \pi - 0.1 - 6$. As you do so, observe that $\mathbf{n}$ will make nearly a complete path along a closed curve but it will change the direction it is pointing from where it started to where it ended.

EXERCISE 2.138. Describe the u-parameter and v-parameter curves on the Enneper surface.

EXERCISE 2.139. In Exploration 2.11(c), you proved the largest value of r for which the Enneper surface has no self-intersections assuming that the intersection occurs on the x_3-axis. In this exercise, prove the same result without assuming the intersection occurs on the x_3-axis.

EXERCISE 2.140. Compute the coefficients of the first and the second fundamental forms for the Enneper surface whose parametrization is

$$\mathbf{x}(u,v) = \left(u - \frac{1}{3}u^3 + uv^2, v - \frac{1}{3}v^3 + u^2v, u^2 - v^2\right).$$

EXERCISE 2.141. A *CMC (Constant Mean Curvature) surface* is a surface that has the same mean curvature everywhere on the surface. Minimal surfaces are a subset of CMC surfaces. Using *DiffGeomTool* sketch the following surfaces and determine which are CMC surfaces:

(a). $\mathbf{x}(u,v) = (u - v, u + v, 2(u^2 + v^2))$, where $-1 < u < 1$, $-1 < v < 1$;
(b). $\mathbf{x}(u,v) = (\cos u, \sin u, v)$, where $-\pi < u < \pi$, $-2 < v < 2$;
(c). $\mathbf{x}(u,v) = \Big((2 + \cos v)\cos u, (2 + \cos v)\sin u, \sin v\Big)$, where $0 < u, v < 2\pi$;
(d). $\mathbf{x}(u,v) = (\sqrt{1 - u^2}\cos v, \sqrt{1 - u^2}\sin v, u)$, where $-1 < u < 1$, $-\pi < v < \pi$;

Minimal Surfaces

EXERCISE 2.142. Use eq (5) to show that the Enneper surface parametrized by

$$\mathbf{x}(u,v) = \left(u - \frac{1}{3}u^3 + uv^2, v - \frac{1}{3}v^3 + u^2v, u^2 - v^2\right)$$

is a minimal surface.

EXERCISE 2.143. Prove Theorem 2.39 in the special case that the surface of revolution has the parametrization

$$\mathbf{x}(u,v) = \big(f(v)\cos u, f(v)\sin u, v\big).$$

EXERCISE 2.144. An oblique cylinder is a cylinder whose side forms an angle θ with the x_1x_2-plane, where $0 < \theta \le \frac{\pi}{2}$. For a fixed θ it can be parametrized by

$$\mathbf{x}(u,v) = (\cos u, \sin u + v\cos\theta, v\sin\theta).$$

Determine the values of θ for which $\mathbf{x}$ is isothermal.

EXERCISE 2.145. Show that the parametrization

$$\mathbf{x}(u,v) = \Bigg(\arctan\left(\frac{2u}{1-(u^2+v^2)}\right), \arctan\left(\frac{-2v}{1-(u^2+v^2)}\right),$$
$$\frac{1}{2}\ln\left(\frac{(u^2-v^2+1)^2+4u^2v^2}{(u^2-v^2-1)^2+4u^2v^2}\right)\Bigg)$$

is an isothermal parametrization of Scherk's doubly periodic surface (that is, show that it is isothermal and that there is transformation that maps this parametrization to the parametrization given in Exercise 2.43(b) for Scherk's doubly periodic surface).

EXERCISE 2.146. Let X be a minimal surface that is not a plane, and let Y be its conjugate minimal surface. Is it possible that the plane is one of the associated surfaces for X and Y? If so, describe the geometry of the minimal surface X. If not, explain why.

Weierstrass Representation

EXERCISE 2.147. In Example 2.67, show the details in going from the Enneper surface parametrization

$$\mathbf{x} = \left(\mathrm{Re}\left\{z - \frac{1}{3}z^3\right\}, \mathrm{Re}\left\{-i\left(z + \frac{1}{3}z^3\right)\right\}, \mathrm{Re}\left\{z^2\right\}\right)$$

to the parametrization

$$\mathbf{x}(u,v) = \left(u - \frac{1}{3}u^3 + uv^2, v - \frac{1}{3}v^3 + vu^2, u^2 - v^2\right)$$

that is also for the Enneper surface.

Exercise 2.148. Compute the parametrization for the minimal surfaces generated by using $p(z) = \frac{1}{2z}$ and $q(z) = iz$ on the domain $\mathbb{C} - \{0\}$ in the Weierstrass representation. Use *MinSurfTool* with the `W.E.(p,q)` tab to graph an image of this surface which is known as the *wavy plane*. [Use radius min=0.001, radius max=1.3, theta min=-pi, theta max=pi with initial values $x = \mathrm{Re}(1/2 * \log(z) - 1/4 * z^2)$, $y = \mathrm{Im}(1/2 * \log(z) + 1/4 * z^2)$, and $z = \mathrm{Re}(z)$.]

Exercise 2.149. Compute the parametrization for the minimal surfaces generated by using $p(z) = z^2$ and $q(z) = \frac{i}{z^2}$ on the domain $\mathbb{C} - \{0\}$ in the Weierstrass representation. Use *MinSurfTool* with the `W.E.(p,q)` tab to graph an image of this surface which is known as *Richmond's surface*. [Use radius min=0.1, radius max=1, theta min=pi/24, theta max=2pi+pi/24 with initial values $x = \mathrm{Re}(1/3 * z^3 + 1/z)$, $y = \mathrm{Im}(1/3 * z^3 - 1/z)$, and $z = \mathrm{Re}(2 * z)$.]

Exercise 2.150. Compute the parametrization for the minimal surfaces generated by using $p(z) = \frac{(z+1)^2}{z^4}$ and $q(z) = \frac{z^2(z-1)}{z+1}$ on the domain $\mathbb{D} - \{0\}$ in the Weierstrass representation. Use *MinSurfTool* with the `W.E.(p,q)` tab to graph an image of this surface which is known as the *wavy plane*. [Use radius min=0.1, radius max=0.9, theta min=pi/24, theta max=2pi+pi/24 with initial values $x =$, $y =$, and $z =$.]

The Gauss map, G, and height differential, dh

Exercise 2.151. Show that if $z = x + iy$ is the projection of the point (x_1, x_2, x_3) on the Riemann sphere onto to complex plane, then

$$x = \frac{x_1}{1 - x_3}, \quad y = \frac{x_2}{1 - x_3}.$$

Exercise 2.152. For Scherk's doubly periodic surface find:

(a) $G(0)$; (b) $G(1)$; (c) $G(-1)$; (d) $G(i)$; (e) $G(-i)$.

Exercise 2.153. The Weierstrass data for a 4-noid are

$$G(z) = z^3 \quad \text{and} \quad dh = \frac{z^3}{(z^4-1)^2}\, dz.$$

Show that the ends of the 4-noid are catenoid ends.

Exercise 2.154. Determine the asymptotic and curvature lines for Scherk's doubly periodic surface with $G(z) = z$ and $dh(z) = \frac{iz}{z^4-1}dz$.

Exercise 2.155. Determine the period conditions for the wavy plane with $G(z) = z$ and $dh(z) = dz$.

Exercise 2.156. Let M be the Scherk doubly periodic surface with 6 ends. Using the approach of Example 2.104 determine G and dh for this surface.

Minimal Surfaces and Harmonic Univalent Mappings

EXERCISE 2.157. Prove that if $f = u+iv$ is harmonic in a simply-connected domain G, then $f = h + \overline{g}$, where h and g are analytic.

EXERCISE 2.158. Prove that the representations $f(z) = h(z) + \overline{g}(z)$ and $f(z) = \operatorname{Re}\left\{h(z) + g(z)\right\} + i \operatorname{Im}\left\{h(z) - g(z)\right\}$ are equivalent.

EXERCISE 2.159. Shear $h(z) - g(z) = \frac{z}{1-z}$ with $\omega(z) = z^2$ to get the harmonic univalent function $f = h + \overline{g}$ given in Example 2.129, where

$$h = \frac{1}{8}\log\left(\frac{z+1}{z-1}\right) + \frac{3z-2z^2}{4(1-z)^2} \quad \text{and} \quad g = \frac{1}{8}\log\left(\frac{z+1}{z-1}\right) - \frac{z-2z^2}{4(1-z)^2}.$$

EXERCISE 2.160. Shear $h(z) - g(z) = \frac{z}{(1-z)^2}$ with $\omega(z) = z^2$ to get the harmonic univalent function $f = h + \overline{g}$ given in Exercise 2.130, where

$$h = \frac{1}{8}\log\left(\frac{z+1}{z-1}\right) + \frac{3z-2z^2}{4(1-z)^2} \quad \text{and} \quad g = \frac{1}{8}\log\left(\frac{z+1}{z-1}\right) - \frac{z-2z^2}{4(1-z)^2}.$$

EXERCISE 2.161. Show that the parametrization:

$$\mathbf{x} = \left(\operatorname{Re}\left[\frac{i}{2}\log\left(\frac{i+z}{i-z}\right)\right], \operatorname{Im}\left[\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right], \frac{1}{2}\operatorname{Im}\left[i\log\left(\frac{1+z^2}{1-z^2}\right)\right]\right)$$

is equivalent to the parametrization in Exercise 2.145 that gives Scherk's doubly periodic minimal surface.

EXERCISE 2.162. Consider the harmonic univalent map $f(z) = h(z) + \overline{g}(z)$, where

$$\begin{aligned} h =& \frac{3}{16}\log\left(\frac{1+z}{1-z}\right) - \frac{3i}{16}\log\left(\frac{1+iz}{1-iz}\right) + \frac{1}{4}\frac{z}{1-z^4} \\ g =& -\frac{3}{16}\log\left(\frac{1+z}{1-z}\right) - \frac{3i}{16}\log\left(\frac{1+iz}{1-iz}\right) + \frac{1}{4}\frac{z^3}{1-z^4}. \end{aligned}$$

(a) Use Theorem 2.124 to find the parametrization of the minimal graph that f lifts to.
(b) Use *ComplexTool* to graph the image of $\mathbb{D}$ under f and *MinSurfTool* with tab **W.E.(h,g)** to sketch the corresponding minimal graph.
(c) This minimal surface has 4 helicoidial ends and is not embedded (i.e., it does have self-intersections). However, Theorem 2.124 states that this surface should be a minimal graph and hence have no self-intersections. Explain why this is not a contradiction.

LARGE PROJECT 2.163. The analytic function, $F(z) = \frac{z}{(1-z)^2}$, maps the unit disk, $\mathbb{D}$, onto $\mathbb{C} \setminus (-\infty, -\frac{1}{4})$ and is an important function. Shear $h(z) - g(z) = \frac{z}{(1-z)^2}$ with various perfect-square dilatations, ω, that satisfy the condition $|\omega| < 1$ for all $z \in \mathbb{D}$ (e.g., $\omega = z^{2n}(n = \mathbb{N})\omega = e^{\imath\theta}z^2, (\theta \in \mathbb{R}), \omega = \left(\frac{z-a}{1-\bar{a}z}\right)(|a| < 1)$). Determine the corresponding minimal graphs.

Bibliography

[1] F. J. Almgren and J. Taylor, The geometry of soap films and soap bubbles, *Sci. Am.* **235** (1976), 8293.

[2] L. Bers, *Riemann Surfaces*, New York Univ., Institute of Mathematical Sciences, New York, 1957-1958.

[3] W. Boothby, *An introduction to differentiable manifolds and Riemannian geometry*, Second edition. Pure and Applied Mathematics, vol 120. Academic Press, Inc., Orlando, FL, 1986.

[4] J. Clunie and T. Sheil-Small, Harmonic univalent functions, *Ann. Acad. Sci. Fenn. Ser. A.I Math.* **9** (1984), 3-25.

[5] R. Courant and H. Robbins, *What Is Mathematics?*, Oxford University Press, 1941.

[6] U. Dierkes, S. Hildebrandt, A. Kster, and O. Wohlrab, *Minimal surfaces I*, Grundlehren der Mathematischen Wissenschaften, 295, Springer-Verlag, Berlin, 1992.

[7] M. do Carmo, *Differential geometry of curves and surfaces*, translated from the Portuguese, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1976.

[8] M. Dorff, Minimal graphs in $\mathbb{R}^3$ over convex domains, *Proc. Amer. Math. Soc.* **132** (2004), no. 2, 491-498.

[9] M. Dorff and J. Szynal, Harmonic shears of elliptic integrals, *Rocky Mountain J. Math.* **35** (2005), no. 2, 485–499.

[10] K. Driver and P. Duren, Harmonic shears of regular polygons by hypergeometric functions, *J. Math. Anal. App.* **239** (1999), 72-84.

[11] P. Duren and W. Thygerson, Harmonic mappings related to Scherk's saddle-tower minimal surfaces, *Rocky Mountain J. Math.* **30** (2000), no. 2, 555-564.

[12] W. Hengartner and G. Schober, On schlicht mappings to domains convex in one direction, *Comment. Math. Helv.* **45** (1970), 303-314.

[13] S. Hildebrandt and A. Tromba, *Mathematics and Optimal Form*, Scientific American Library, 1985.

[14] H. Karcher, Construction of minimal surfaces, *Survey in Geometry*, Univ. of Tokyo, 1989.

[15] H. Karcher, Introduction to the complex analysis of minimal surfaces, Lecture notes given at the NCTS, Taiwan, 2003.

[16] H. Lewy, On the non-vanishing of the Jacobian in certain one-to-one mappings, *Bull. Amer. Math. Soc.* **42** (1936), no. 10, 689-692.

[17] J. McDougall and L. Schaubroeck, Minimal surfaces over stars, *J. Math. Anal. Appl.* **340** (2008), no. 1, 721-738.

[18] R. Millman and G. Parker, *Elements of differential geometry*, Prentice-Hall Inc., Englewood Cliffs, N. J., 1977.

[19] F. Morgan, Minimal surfaces, crystals, shortest networks, and undergraduate research, *Math. Intel.*, **14** (1992), 37-44.

[20] J. Nitsche, *Lectures on Minimal Surfaces*, vol. 1, Cambridge U. Press, 1989.

[21] J. Oprea, *Differential geometry and its applications.*, 2nd ed., Classroom Resource Materials Series. Math. Assoc. of America, Washington, DC, 2007.

[22] J. Oprea, *The mathematics of soap films: explorations with Maple*, Student Mathematical Library, vol 10, Amer. Math. Soc., Providence, RI, 2000.
[23] R. Osserman, *A survey of minimal surfaces*, 2nd ed. Dover Publications, Inc., New York, 1986.
[24] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, vols 1-5, 3rd ed., Publish or Perish, Inc., Houston, TX, 2005.
[25] M. Weber, *Classical minimal surfaces in Euclidean space by examples*, Preliminary notes for the Clay Institute Summer School on minimal surfaces, MSRI, Berkeley, Calif., 2001.
[26] S. Weintraub, *Differential Forms: A Complement to Vector Calculus*, Academic Press, Inc., San Diego, CA, 1997.