APPENDIX A

Background

Notation Page

The following notation will be defined and employed throughout the text, and so we collect it here for easy reference.

$\mathbb{N}$ = the natural numbers.
$\mathbb{Q}$ = the rational numbers.
$\mathbb{R}$ = the real numbers.
$\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ = the extended real numbers.
$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ = the Euclidean plane.
$\mathbb{C} = \{x + iy : x, y \in \mathbb{R}; i = \sqrt{-1}\}$ = the complex numbers.
$\mathrm{Re}(z)$ = the real part of z.
$\mathrm{Im}(z)$ = the imaginary part of z.
$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ = the extended Complex plane or Riemann Sphere.
$C(a, r) = \{z \in \mathbb{C} : |z - a| = r\}$, Euclidean circle.
$\triangle(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$, open Euclidean disk.
$\mathbb{D} = \triangle(0, 1)$ = the open unit disk centered at 0.
σ = the spherical metric on $\overline{\mathbb{C}}$.
$\triangle_\sigma(a, r) = \{z \in \overline{\mathbb{C}} : \sigma(z, a) < r\}$, open spherical disk.
$\mathrm{Int}(E)$= interior of a set E in $\overline{\mathbb{C}}$.
$\overline{E} = closure(E)$ = closure of a set E in $\overline{\mathbb{C}}$.
∂E = boundary of a set E in $\overline{\mathbb{C}}$.
$domain(f)$ = domain set of function f.
$\mathrm{Res}(f, z_0)$ = the residue of f at z_0.

This appendix is intended to be a summary of some of the major topics and theorems from a standard undergraduate complex analysis course. Since it is only a review of the material, we do not prove all the results here, but rather direct readers looking for more details to the main reference texts [**1**] and [**3**], and occasionally to the more advanced text [**2**]. Additionally, a few advanced topics are also included here. These include the Riemann Mapping Theorem, the Open Mapping Theorem, and the Schwarz Lemma. The extended notion of analyticity on the Riemann sphere $\overline{\mathbb{C}}$ is addressed separately in Appendix B.

A.1. Functions of a Complex Variable as Mappings

A complex valued function f defined on a set $R \subset \mathbb{C}$ is denoted $f : R \to \mathbb{C}$. We often denote the domain set of f by *domain*(f), and without further mention use the convention that we may express $z = x+iy = (x, y)$ and $f(z) = f(x, y)$ with the implicit understanding that $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$. Also, the set $f(R) = \{f(z) : z \in R\}$ is called the *range* of f. Lastly, if $A \subset \mathbb{C}$, we denote the *inverse image* (or preimage) of A under f by $f^{-1}(A) = \{z \in R : f(z) \in A\}$.

A.1.1. Linear Functions. We can examine the linear mappings $f(z) = az+b$, for constants $a, b \in \mathbb{C}$, by first considering the simpler functions $h(z) = az$ and $g(z) = z+b$.

Writing the variable $z = re^{i\theta}$ and the parameter $a = |a|e^{i\alpha}$ in polar form, we can express the image point as $h(z) = az = r|a|e^{i(\theta+\alpha)}$. Hence, the function h maps the point z to the point $h(z)$, which geometrically can be interpreted as a stretching or contraction of the modulus by $|a|$ and a rotation about the origin by the angle α.

We can understand the action of the function $g(z) = z + b$ geometrically as translating the point z by $|b|$ units in the direction of the *vector* b.

Putting this together we see that the function $f(z) = az + b$ is the composition $g(h(z))$ which moves a point z by stretching or contracting by $|a|$, rotating by the angle α, and then translating by the point (vector) b. See Figure A.1 for an example.

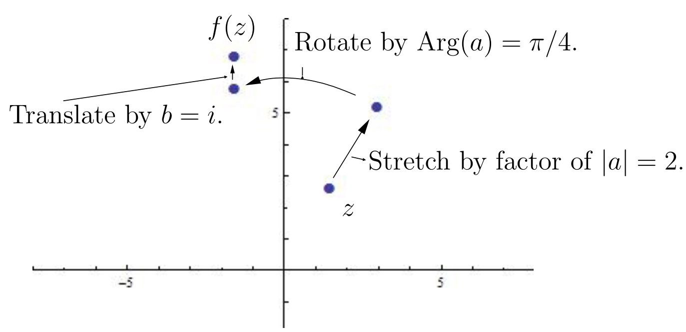

FIGURE A.1. An illustration of the map $f(z) = az + b$ for $a = 2e^{i\pi/4}$ and $b = i$.

A.1.2. Power Functions. Consider the map $f(z) = z^n$ for fixed $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the natural numbers. Expressing the variable $z = re^{i\theta}$ in polar form, we see that $f(z) = (re^{i\theta})^n = r^n e^{in\theta}$; that is, under the action of f the point z has its modulus raised to the nth power and its argument multiplied by n. See Figure A.2 for an illustrative example.

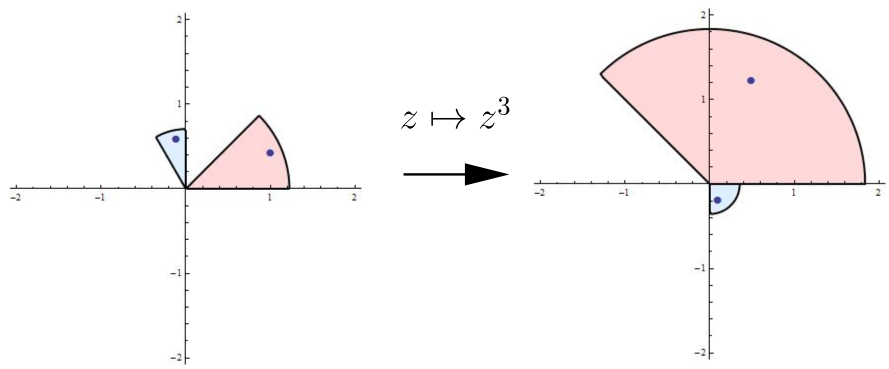

FIGURE A.2. An illustration of the map $z \mapsto z^3$.

A.1.2.1. *Roots of unity.* Fix a positive integer n. We call any of the n solutions to the equation $z^n = 1$ an *nth root of unity*, noting that each must be of the form $\omega_k = e^{2k\pi i/n}$ for $k = 0, \ldots, n-1$. We also note that these nth roots of unity can be expressed $1, \omega_1, \omega_1^2, \ldots, \omega_1^{n-1}$ and that these are equally spaced points on the unit circle $|z| = 1$.

A.1.3. The Exponential Function. The definition of the exponential function arises naturally out of Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.

DEFINITION A.1. If $z = x + iy$, then

$$e^z = e^x(\cos y + i \sin y).$$

The preceding definition has two important consequences:

(1) The exponential function maps $\mathbb{C}$ onto $\mathbb{C}\backslash\{0\}$.
(2) The exponential function is periodic of period $2\pi i$.

A.1.4. The Logarithm Function. The essence of the logarithm function is in its role in providing an inverse of the exponential function. However, since the exponential function is not one-to-one, special consideration must be taken by first understanding $\arg(z)$.

DEFINITION A.2. For a complex number $z \neq 0$, the argument of z, denoted $\arg(z)$, is the set of all $\theta \in \mathbb{R}$ such that $z = |z|e^{i\theta}$. The *principal value* of the argument, denoted $\operatorname{Arg}(z)$, is the unique such angle θ with $-\pi < \theta \leq \pi$.

Definition A.3. The function $\log(z)$ is the multiple-valued function

$$\log(re^{i\theta}) = \ln(r) + i\theta,$$

where $\ln(r)$ denotes the real-valued logarithm. Alternatively, we may define

$$\log(z) = \ln|z| + i\arg(z).$$

Since $\log(z)$ is multiple-valued (because $\arg(z)$ is multiple-valued), it is not really a **function** in the traditional sense. To make it a function, we can use the *principal value* of the argument, $\mathrm{Arg}(z)$.

Definition A.4. The *principal value* of the logarithm is the function defined by

$$w = \mathrm{Log}(z) = \ln|z| + i\mathrm{Arg(z)}.$$

It has domain set $\mathbb{C}\backslash\{0\}$ and range $-\pi < \mathrm{Im}(w) \leq \pi$, but is not continuous at any point of the negative real axis.

Important Technology Note: While mathematically we use $\mathrm{Log}(z)$ and $\mathrm{Arg}(z)$ to denote the principal values of the logarithm and argument functions, the applet *ComplexTool* uses $\log(z)$ and $\arg(z)$ to express these.

Remark A.5. While the principal branch of the logarithm is the one most widely used, it is possible to define other branches of the logarithm function which are continuous, for example, along the negative real axis. Note that any branch of the logarithm must necessarily include 0 as a point of discontinuity. For more information, we refer the reader to [**1**] and [**3**].

A.1.5. Trigonometric Functions. The trigonometric functions of a complex number z are defined in terms of the exponential function.

Definition A.6. Given any complex number z, we define

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \text{and} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

One can show that the image of the infinite vertical strip $-\pi/2 \leq \mathrm{Re}\, z \leq \pi/2$ under either of these maps is the entire plane. (Thus the trigonometric functions are unbounded in $\mathbb{C}$.)

A.2. Continuity and Analyticity in $\mathbb{C}$

Definition A.7. Let $R \subset \mathbb{C}$ and consider a complex valued function $f : R \to \mathbb{C}$.

(1) We say that f is *continuous* at z_0 if $\lim_{z\to z_0} f(z) = f(z_0)$; i.e., for each $\epsilon > 0$ there exists $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ whenever $z \in R$ and $|z - z_0| < \delta$. We say f is continuous on a set $U \subset R$ if it is continuous at each point of U.

(2) Furthermore, f is called *differentiable* at z_0 when its derivative $f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists.

(3) When such a function is differentiable at all points of an *open* set R, the function f is said to be *analytic* on R. (Some authors refer to such a function as *holomorphic* in R.)

A.2.1. Cauchy-Riemann Equations. When we write $f(z) = f(x, y)$, we call $u(x, y) = \operatorname{Re} f(x, y)$ and $v(x, y) = \operatorname{Im} f(x, y)$. Then $f(x, y) = u(x, y) + iv(x, y)$ and thus we may define the following partial derivatives

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}. \tag{100}$$

We recall that when f is differentiable at z_0, we have $f'(z_0) = \frac{\partial f}{\partial x}(z_0) = -i\frac{\partial f}{\partial y}(z_0)$, which by equating real and imaginary parts yields the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{101}$$

THEOREM A.8. A function $f(z) = u(x, y) + iv(x, y)$ is analytic in an open set R if and only if the first partial derivatives u_x, u_y, v_x, v_y exist, are continuous, and satisfy the Cauchy-Riemann equations in R. (See, e.g., [**1**], p. 63-66.)

Furthermore, by using the Cauchy-Riemann equations (101), one can show that if $f(z) = u(x, y) + iv(x, y)$ is analytic in an open set R, then each of the component functions u and v is *harmonic* in R, that is, $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$.

A.2.2. Conformal Mappings. Recall that a function $f : R \to \mathbb{C}$ is called *univalent* if it is one-to-one; i.e., for any two points $z_1 \neq z_2$ in R, we then have $f(z_1) \neq f(z_2)$. An analytic function is **locally** univalent in a small neighborhood of a point z_0 if and only if its derivative is non-zero at z_0. A function that is both analytic and univalent on an open set is said to be *conformal* on that set. Geometrically, conformal means that the function is locally angle-preserving, preserving both the magnitude and sense of the angle. You can get a feel for this by graphing any univalent analytic function with the applet *ComplexTool*, using either a rectangular or circular grid, and zooming in on points in the range to see the angle preservation.

A.3. Complex Integration

A complex integral is an expression of the form $\int_C f(z)\, dz$. This integral can be evaluated for many kinds of functions (analytic or not) and any path (closed loop or not). In many cases, the value of the integral does depend on the actual geometry of the path, not just on the end points. It is important to note that the value of the integral does not depend on the parameterization used to describe the path as long as the different parameterizations trace out the path in the same direction. This will be evident in the formula given below.

A.3.1. Computing a Complex Integral. To compute a complex integral by brute force requires some parameterization of the path C. We describe the path directly in terms of z whenever possible. Here are some important examples of curves and how they could be parameterized:

(1) A circle of radius R centered at a: $z(t) = a + R\,e^{it}$, where $0 \le t \le 2\pi$.
(2) The line segment from z_1 to z_2: $z(t) = (1-t)z_1 + tz_2$, where $0 \le t \le 1$.

Once a parameterization is obtained, the formula for calculating the integral is

$$\int_C f(z)\,dz = \int_{t=a}^{b} f(z(t))z'(t)\,dt. \tag{102}$$

A.3.2. Topology on the plane. [1]

Definition A.9.

(1) A set is a *domain* if it is open and connected.
(2) A path C is a *closed curve* if its initial and terminal points coincide.
(3) A *simple closed curve* is a closed curve that does not cross itself.

A.3.3. Three Important Integral Theorems. The following results are central to any study of complex analysis.

Theorem A.10 (Cauchy's Theorem). Let $f(z)$ be analytic *everywhere on and inside* a simple closed curve γ. Then

$$\int_\gamma f(z)\,dz = 0.$$

Theorem A.11 (Cauchy's Integral Formula). Let $f(z)$ be analytic *everywhere on and inside* a simple closed positively-oriented curve γ. Let a be a point inside γ. Then

$$\int_\gamma \frac{f(z)}{z-a}\,dz = 2\pi\,i\,f(a).$$

Theorem A.12 (Cauchy's Generalized Integral Formula). Let $f(z)$ be analytic *everywhere on and inside* a simple closed positively-oriented curve γ. Let a be a point inside γ. Then

$$\int_\gamma \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi\,i\,f^{(n)}(a)}{n!}.$$

[1] See Appendix B.3 for more detailed information regarding topology on both the plane and the Riemann sphere.

A.3.4. Path Integrals. If C is not a simple closed curve, then none of the above integral theorems can be used to calculate it. Fortunately it may not be necessary to use brute-force either. Suppose that C is a path that goes from a point z_1 to the point z_2. The main question is when does the value of the integral only depend on the endpoints and not on the particular path between them? The answer is exactly when $f(z)$ is analytic on an appropriate region. In this case the integral can be calculated by finding an anti-derivative and evaluating it at the endpoints. This is simply an application of the Fundamental Theorem of Calculus.

Here is a more precise statement: If C_1 and C_2 are paths between the same endpoints, then $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$ if $f(z)$ is *analytic everywhere on and in between both paths.*

A.4. Taylor Series and Laurent Series

Taylor series and Laurent series play crucial roles in our understanding of analytic functions.

A.4.1. Taylor Series. The theory of Taylor series carries over directly from the theory in real variables. However, it is even better as the following theorem shows:

THEOREM A.13. Let $f(z)$ be an *analytic function* in a domain D and let $z_0 \in D$. Then the following statements hold:

(1) $f(z)$ can be represented by a convergent power series in the disk $|z-z_0| < R$, where R is the distance from z_0 to the nearest singularity of $f(z)$. In particular, if $f(z)$ is entire, then $R = \infty$ and the series converges everywhere.
(2) The series representing $f(z)$ is the Taylor series; hence

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n.$$

(3) The Taylor series is unique, in other words, any power series representing an analytic function must in fact be the Taylor series of the function.

Note that power series can be added, subtracted, multiplied and divided. Caution: multiplication and division are not performed term by term; rather you must treat each series like a polynomial. Series can also be differentiated and integrated term-by-term to obtain new series.

A.4.2. Laurent Series. If a complex function has a singularity at $z = z_0$, then it certainly does not have a Taylor expansion at that point. It may, however, have a different sort of series expansion in a deleted neighborhood of the singularity called a *Laurent series.*

In general, a function f which is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$ has a unique Laurent series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

on that annular domain. It is allowed that we may have $R_1 = 0$ or $R_2 = +\infty$, or both.

We note that just as in the case of Taylor series, there are formulas that give the coefficients of the desired Laurent series. However, these coefficient formulas involve complicated complex integrals and are of more theoretical than practical value.

One of the main values of the Laurent expansion stems from its use in complex integration via residues. When f can be expressed by the Laurent series $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ on a punctured disk with center z_0, the *residue* of f at z_0 is $\operatorname{Res}(f, z_0) = b_1$.

Theorem A.14 (Residue Theorem). Let γ be a simple closed positively-oriented curve. If a function f is analytic inside and on γ except for a finite number of singular points z_k, for $k = 1, \ldots, n$ inside C, then $\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f, z_k)$.

A.4.3. Isolated Singularities. Given a function f which is analytic on a deleted ϵ neighborhood of z_0, we can express f by its Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

on $\Delta(z, \epsilon) \setminus \{z_0\}$. This situation gives us three different types of singularities classified as follows.

Definition A.15 (Isolated singularities).

(i) When $b_n = 0$ for every $n = 1, 2, \ldots$, then $f(z)$ is said to have a *removable singularity* at z_0. In such a case $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, and so defining $f(z_0) = a_0$ will make f analytic in the full neighborhood $\Delta(z, \epsilon)$ (and thus we have *removed* the singularity).

(ii) When only finitely many $b_n \neq 0$, then there exists N such that $b_N \neq 0$ and $b_n = 0$ for all $n > N$. In this case, we say f has a *pole* of order N at z_0. We may then express f in a factored form $f(z) = (z - z_0)^{-N} h(z)$ where $h(z) = b_N + b_{N-1}(z - z_0) + \ldots$ is analytic on $\Delta(z, \epsilon)$ and $h(z_0) = b_N \neq 0$.

(iii) When infinitely many $b_n \neq 0$, we say f has an *essential singularity* at z_0.

Definition A.16. A function f is *meromorphic* in a domain D if it is analytic at every point of D except possibly at poles.

A.5. Key Theorems

This section records several key results that will be used in the text.

A.5.1. Maximum Modulus Theorem. We recall two key results relating to the maximum modulus of an analytic function. See [**1**], p. 176-178.

Theorem A.17 (Maximum Modulus Theorem). If a function f is analytic and not constant in a given domain $D \subset \mathbb{C}$, then $|f(z)|$ has no maximum value in D. That is, there is no point $z_0 \in D$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in D$.

Corollary A.18 (Corollary to Maximum Modulus Theorem). Suppose that a function f is continuous on a closed and bounded region $R \subset \mathbb{C}$ and that it is analytic and not constant in the interior of R. Then, the maximum value of $|f(z)|$ in R, which is always reached, occurs somewhere on the boundary of R and never in the interior.

A.5.2. Argument Principle. The Argument Principle for analytic functions gives a very nice way to count the number of zeros and poles inside a contour.

Theorem A.19 (Argument Principle). Let γ be a simple closed curve lying entirely within a domain $D \subset \mathbb{C}$. Suppose f is analytic in D except at a finite number of poles inside γ and that $f(z) \neq 0$ on γ. Then

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}dz = N_0 - N_p,$$

where N_0 is the total number of zeros of f inside γ and N_p is the total number of poles of f inside γ. In determining N_0 and N_p, zeros and poles are counted according to their order or multiplicities.

A.5.3. Schwarz Lemma.

Theorem A.20 (Schwarz Lemma). Suppose $f : \mathbb{D} \to \overline{\mathbb{D}}$ is analytic and $f(0) = 0$. Then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$. Furthermore, unless f is of the form $f(z) = e^{i\theta}z$ for some $\theta \in \mathbb{R}$ (i.e., f is a rotation), we must have strict inequalities in both statements above.

A.6. More Advanced Results

A.6.1. Local Properties of Analytic Maps. The existence of power series representations of analytic maps is a powerful tool for understanding their local, and sometimes global, behavior. Consider an analytic function f defined in a domain $D \subset \mathbb{C}$. For any $z_0 \in D$, Theorem A.13 tells us that we may express

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots \tag{103}$$

on any neighborhood $\Delta(z_0, r)$ which is contained in D.

Suppose for the moment that $f'(z_0) = a_1 \neq 0$. Then for z *very close* to z_0 we see that the linear contribution $a_0 + a_1(z - z_0)$ dominates the rest of the series in (103). Omitting the details, we can say that $f(z) \approx A_1(z) = a_0 + a_1(z - z_0)$ for $z \in \Delta(z_0, \epsilon)$ when $\epsilon > 0$ is very small. Since $A_1(z)$ is a linear map, with very well understood properties, we can reasonably expect that $f(z)$ will have the same properties as $A_1(z)$

when $z \in \Delta(z_0, \epsilon)$. In particular, for every w near $a_0 = f(z_0)$, there is exactly one value z_1 near z_0 which maps to w. This holds for A_1, and it also holds for f. Thus we say that f is *locally* one-to-one (also called locally univalent) at z_0 when $f'(z_0) \neq 0$. See Figure A.3.

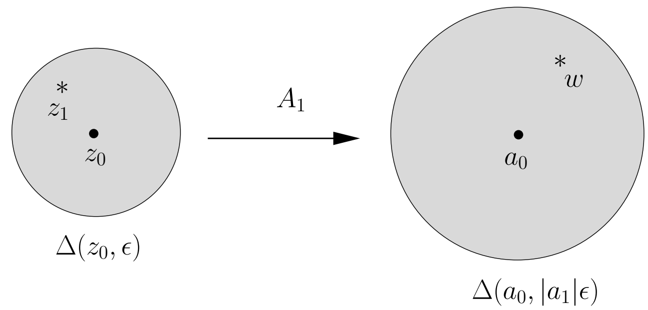

FIGURE A.3. An illustration of the map $A_1(z) = a_0 + a_1(z - z_0)$ mapping $\Delta(z_0, \epsilon)$ onto $\Delta(a_0, |a_1|\epsilon)$ in a one-to-one fashion.

Similarly, we can use approximations to understand the local behavior of $f(z)$ when $a_1 = 0$, i.e., $f'(z_0) = 0$. In such a case, we know that (unless f is constant) we may express f as

$$f(z) = a_0 + a_k(z - z_0)^k + \dots \tag{104}$$

where a_k is the first non-zero coefficient (other than possibly a_0) in (103). Then for z *very close* to z_0, we see that the terms $a_0 + a_k(z - z_0)^k$ dominate the rest of the series in (104). Omitting the details, we can say that $f(z) \approx A_k(z) = a_0 + a_k(z - z_0)^k$ for $z \in \Delta(z_0, \epsilon)$ when $\epsilon > 0$ is very small. Since $A_k(z)$ has very well understood properties,[2] we can reasonably expect that $f(z)$ will have the same properties as $A_k(z)$ when $z \in \Delta(z_0, \epsilon)$. In particular, for every w near $a_0 = f(z_0)$, there are exactly k values $z_1, \dots, z_k$ symmetrically arranged near z_0 which maps to w. This holds for A_k, and it also holds for f, though since we only have an approximation the k values may not be exactly symmetrically arranged about z_0. See Figure A.4. Thus we say that f is *locally* k-to-one at z_0 when $f'(z_0) = f''(z_0) = \cdots = f^{(k-1)}(z_0) = 0$, but $f^{(k)}(z_0) \neq 0$. We also describe this situation by saying that z_0 maps to $f(z_0)$ with ***degree*** (aka, multiplicity or valency) k.

When z_0 maps to $f(z_0)$ with multiplicity k, we can rewrite (104) as $f(z) = a_0 + (z - z_0)^k[a_k + a_{k+1}(z - z_0) + \dots]$. Noting that $a_k + a_{k+1}(z - z_0) + \dots$ determines an analytic map on D, we have the following.

LEMMA A.21. Let f be a function analytic at z_0 with multiplicity k. Then there is a map h which is analytic at z_0 such that $h(z_0) \neq 0$ and $f(z) = f(z_0) + (z - z_0)^k h(z)$.

[2]In particular, we note that $A_k(z)$ is a composition $h_3 \circ h_2 \circ h_1$ of the following simple maps $h_1(z) = z - z_0$ (translation), $h_2(z) = z^k$ (power function), and $h_3(z) = a_0 + a_k z$ (linear function).

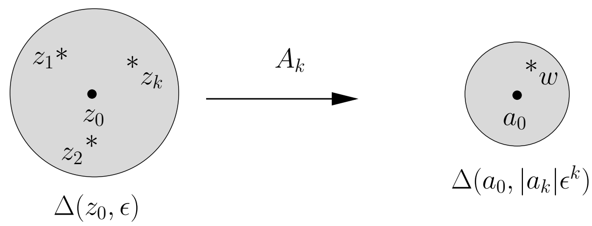

FIGURE A.4. An illustration of the map $A_k(z) = a_0 + a_k(z - z_0)^k$ for $k = 3$ mapping $\Delta(z_0, \epsilon)$ onto $\Delta(a_0, |a_k|\epsilon^k)$ in a k-to-one fashion.

In particular, if $f(z_0) = 0$, i.e., f has a zero of order k at z_0, then f has the form $f(z) = (z - z_0)^k h(z)$.

A.6.1.1. *Open Mapping Theorem.* A fact not usually introduced in a standard undergraduate complex variables course is that analytic maps are open maps, which we define more carefully below. Some chapters in this text reference this and other such results, and so we present them here. The interested reader can find formal proofs in more advanced texts such as [**2**], p. 344-348.

Consider a non-constant function f which is analytic at z_0. The discussion above shows that (whether or not $f'(z_0) = 0$) the image of a small neighborhood of z_0 must cover a small neighborhood of $f(z_0) = a_0$. This is enough to assert the following.

THEOREM A.22 (Open Mapping Theorem). If f is a function on a domain $D \subset \mathbb{C}$ which is analytic and non-constant, then the range $f(D)$ is an open set.

Since ***open maps*** are those for which the image of an open set is always an open set, non-constant analytic maps are open maps.

One application of this and the above theory is the following.

THEOREM A.23 (Inverse Function Theorem). Suppose that $D \subset \mathbb{C}$ is a domain and $f : D \to \mathbb{C}$ is a univalent analytic map. Then its inverse function $f^{-1} : f(D) \to D$ is also analytic.

A nice use for the Inverse Function Theorem is that it shows that *locally* one-to-one functions have local analytic inverse functions. More precisely, consider an analytic map f defined on a domain Ω, where f need not necessarily be one-to-one on all of Ω. If $f'(z_0) \neq 0$ for some $z_0 \in \Omega$, then we know from above that there is a small disk $D = \Delta(z_0, \epsilon)$ on which f is one-to-one. Hence by the Inverse Function Theorem, we see that there is a map $g : f(D) \to D$ (the *local inverse* of f) defined on the open set $f(D)$ which is analytic and satisfies $g \circ f(z) = z$ for all $z \in D$.

A.6.1.2. *Riemann Mapping Theorem.* We close with a very important theorem in complex analysis which states that all simply connected domains (other than the whole plane) are conformally equivalent to the unit disk. More precisely we have the following, whose proof can be found in [**2**], p. 420.

THEOREM A.24 (Riemann Mapping Theorem). Suppose that D is a simply connected domain in the complex plane, $D \neq \mathbb{C}$, and that $z_0 \in D$. Then, there exists a unique conformal mapping f of D onto the unit disk $\triangle(0, 1)$ satisfying the conditions $f(z_0) = 0$ and $f'(z_0) > 0$.

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APPENDIX B

The Riemann Sphere

The main purpose of this appendix is to refresh some of the key ideas about the concept of ∞ in complex analysis. In particular, we discuss the Riemann sphere via stereographic projection, the induced spherical metric σ and corresponding topology, as well as discuss how these notions tie in with the concepts of continuity and analyticity. We do not re-prove all the results here or give a complete exposition of these topics; our goal is to provide just enough background to allow the reader to fully grasp the material in those chapters of this book which require it. More details can be found in the reference texts [**1**], [**3**], and the more advanced text [**2**].

Complex arithmetic actually extends slightly beyond the complex plane. As points z in $\mathbb{C}$ move arbitrarily far away from the origin we informally say that they go to "infinity". We can make that quite precise, while also giving you one answer to the division-by-zero problem: we adjoin a point denoted ∞ to $\mathbb{C}$ to get the *Riemann sphere* (or *extended complex plane*), denoted here as $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

B.1. Stereographic Projection and Spherical Geometry

We model the Riemann sphere $\overline{\mathbb{C}}$ by first identifying points in $\mathbb{C}$ with points in the unit sphere $S = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbb{R}^3$ through what is called *stereographic projection*. See Figure B.1. Label $N = (0, 0, 1)$ the "north pole" on the sphere S. Then for any given point $z = x + iy = (x, y, 0)$ in $\mathbb{C}$, consider the line in $\mathbb{R}^3$, between z and N.

Obviously, this line intersects S at precisely two points, namely N and a second point we call $Z = (x_1, x_2, x_3)$. We then associate the points z and Z. Formally, we define a map $\pi : \mathbb{C} \to S$ given by $\pi(z) = Z$. Omitting the details (which can be found in [**2**], p. 351), the precise formula for this map is $Z = \pi(z) = \pi(x + iy) = \left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right)$. However, we note that we rarely make use of this formula since it is the idea and the picture that provide the necessary understanding we require.

Note that the only point on S which is not associated with any complex number is the north pole N and that this association between $\mathbb{C}$ and $S \setminus \{N\}$ is bijective.[1] We would like to now decide what meaning there could be to associating a point to N.

Notice that if $|z|$ is very large (i.e., z is very far from the origin), then the corresponding point $Z \in S$ is very close to N. It is then natural to say that any point that

[1] Recall that a map is bijective when it is both one-to-one and onto.

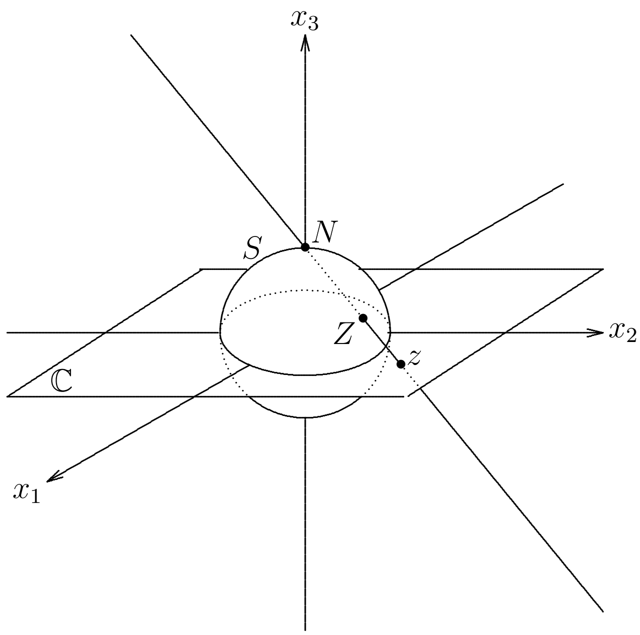

FIGURE B.1. Stereographic projection.

is associated with N must be "infinitely" far from the origin. It is for this reason that we adjoin ∞ to $\mathbb{C}$ and extend the definition of π by defining $\pi(\infty) = N$. We now have an identification, i.e., a bijection, between *all* of $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and *all* of S. Thus when we speak of the Riemann sphere $\overline{\mathbb{C}}$ we can think of it as the usual complex plane together with the point ∞ being "infinitely" far from the origin, or we can think of it as being the associated sphere S where ∞ is understood to be the north pole N. It will be useful to think of $\overline{\mathbb{C}}$ in both these ways; let us do so with some examples.

The origin in $0 \in \mathbb{C}$ is identified with the "south pole" $(0, 0, -1)$, unit disk $\mathbb{D} = \{z : |z| < 1\}$ is identified with the "southern hemisphere", the unit circle $|z| = 1$ is the "equator", and the real axis is wrapped around the sphere S to form a circle running through the the south pole $(0, 0, -1)$, the north pole N, and the points $(1, 0, 0)$ and $(-1, 0, 0)$. Notice that as the real number x_1 goes to $-\infty$ and the real number x_2 goes to $+\infty$, the corresponding points $\pi(x_1)$ and $\pi(x_2)$ on the sphere S come together at the north pole N. So we see that in S there is only one ∞.

As an exercise you are asked to match up the curves in $\mathbb{C}$ with their projections onto $\overline{\mathbb{C}}$ given in Figure B.2.

B.2. The Spherical Metric σ

The standard way we measure the distance between two points z and w in $\mathbb{C}$ is by $|z - w|$, the Euclidean metric on $\mathbb{C}$. But can we come up with a natural way to define the distance between points on $\overline{\mathbb{C}}$ by using our understanding of the sphere model?

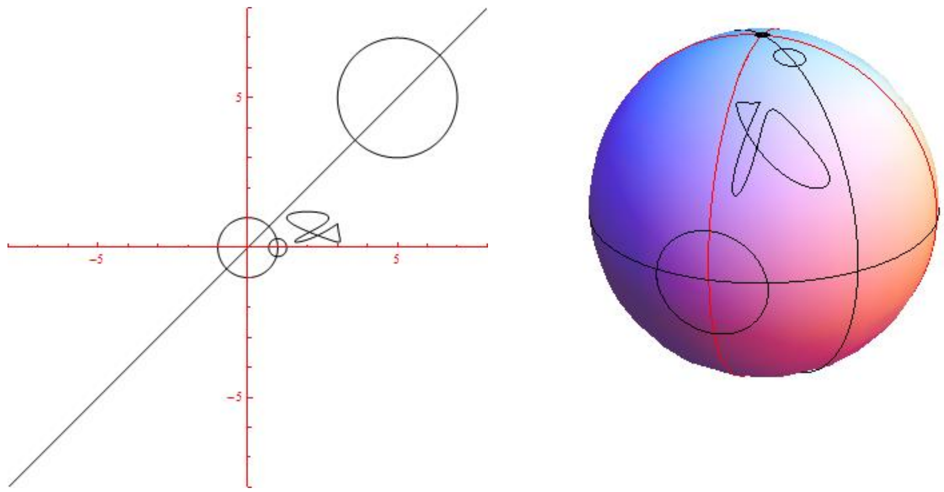

FIGURE B.2. Various curves drawn in the plane and projected onto the sphere, where ∞ is the marked point on top of the sphere $\overline{\mathbb{C}}$.

The answer is yes, we just need to use the natural spherical metric on S and transfer that back to $\overline{\mathbb{C}}$. Here's how.

The spherical distance between two points Z and W on S is defined to be the arclength of the shortest path on the sphere S which connects Z and W, which is, of course, the shorter arc of the great circle that runs through Z and W. We denote this distance by $d(Z, W)$. For example, the distance between $(1, 0, 0)$ and $N = (0, 0, 1)$ is the number $\pi/2$ since this is just $1/4$ of the circumference of a great circle with radius 1.

We can now transfer this metric from S to $\overline{\mathbb{C}}$ using the map $\pi : \overline{\mathbb{C}} \to S$ by defining the distance between two points $z, w \in \overline{\mathbb{C}}$ to be $\sigma(z, w) = d(Z, W) = d(\pi(z), \pi(w))$. This simply amounts to projecting z and w onto the sphere S and then taking the distance there. There does exist a formula for $\sigma(z, w)$, but we omit it since it is not important to our purposes. We will, however, note that this new *metric* (aka distance function) on $\overline{\mathbb{C}}$ does not treat ∞ as a special point. It simply plays the same role as any other point on the Riemann Sphere. This idea can take some getting used to, so let's look at some examples.

We first would like to understand what a *Spherical* disk $\triangle_\sigma(a, r) = \{z \in \mathbb{C} : \sigma(z, a) < r\}$, with center $a \in \overline{\mathbb{C}}$ and radius $r > 0$ looks like. Let's consider $\triangle_\sigma(\infty, \pi/2)$, i.e., the set of all points $z \in \overline{\mathbb{C}}$ such that $\sigma(z, \infty) < \pi/2$. By picturing these points on the sphere we see that the answer is the upper hemisphere, which we note can also be expressed as $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$, i.e., the points in $\mathbb{C}$ outside of the closed unit disk together with ∞. In fact, denoting the *Euclidean* disk of radius $r > 0$ and center $a \in \mathbb{C}$ by $\triangle(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$, we have the following more general result. For any small $\epsilon > 0$, there is some large number $r > 0$ such that

$$\Delta_\sigma(\infty, \epsilon) = \overline{\mathbb{C}} \setminus \overline{\Delta(0, r)}. \tag{105}$$

B.3. Topology in $\mathbb{C}$ and $\overline{\mathbb{C}}$

With the spherical metric in hand, we can now define the corresponding topological concepts on $\overline{\mathbb{C}}$. We begin by defining the interior, closure, and boundary of a set $E \subset \overline{\mathbb{C}}$ as follows: The ***interior*** of E, denoted $\mathrm{Int}(E)$, is the set which contains all points $z \in E$ for which there exists $r > 0$ such that $\Delta_\sigma(z, r) \subset E$. The ***closure*** of E, denoted $\overline{E}$ or $closure(E)$, is the set which contains all points $z \in \overline{\mathbb{C}}$ such that for any $r > 0$ we have $\triangle_\sigma(z, r) \cap E \neq \emptyset$. The ***boundary*** of E, denoted ∂E, is the set which contains all points $z \in \overline{\mathbb{C}}$ such that for any $r > 0$ we have both $\Delta_\sigma(z, r) \cap E \neq \emptyset$ and $\Delta_\sigma(z, r) \cap (\overline{\mathbb{C}} \setminus E) \neq \emptyset$. Hence, $\partial E = \overline{E} \cap \overline{\overline{\mathbb{C}} \setminus E}$ and $\overline{E} = E \cup \partial E$.

Example B.1. Without needing to provide a formal proof, the readers should convince themselves that $\mathrm{Int}(\mathbb{D}) = \mathbb{D}, \overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$, and $\partial \mathbb{D}$ is the unit circle $|z| = 1$. In general, $\partial \triangle(a, r) = C(a, r)$, where we define $C(a, r) = \{z \in \mathbb{C} : |z - a| = r\}$, i.e., the *Euclidean* circle of radius $r > 0$ and center $a \in \mathbb{C}$.

Exercise B.2. Identify the interior, closure, and boundary of each of the following subsets of $\overline{\mathbb{C}}$. No proof is required. ***Try it out!***

(a) $A = \emptyset$,
(b) $B = \{z \in \mathbb{C} : |z| \leq 1\}$,
(c) $C = \{z \in \mathbb{C} : |z| = 1\}$,
(d) $D = \mathbb{C} \setminus \mathbb{R}$,
(e) $E = \{x + iy \in \mathbb{C} : x, y \in \mathbb{Q}\}$, where $\mathbb{Q}$ is the set of rational numbers in $\mathbb{R}$,
(f) $F = \{1/n : n \in \mathbb{N}\}$,
(g) $G = \mathbb{C}$.

Definition B.3 (Open sets).

(a) A set $A \subset \overline{\mathbb{C}}$ is called *open in* $\overline{\mathbb{C}}$ if $\mathrm{Int}(A) = A$, i.e., if for each point $z \in A$, there exist $r > 0$ such that $\triangle_\sigma(z, r) \subset A$.
(b) A set $A \subset \mathbb{C}$ is called *open in* $\mathbb{C}$ if for each point $z \in A$, there exist $r > 0$ such that $\triangle(z, r) \subset A$.

Definition B.4 (Closed sets).

(a) A set $A \subset \overline{\mathbb{C}}$ is called *closed in* $\overline{\mathbb{C}}$ if its complement $\overline{\mathbb{C}} \setminus A$ is open in $\overline{\mathbb{C}}$.
(b) A set $A \subset \mathbb{C}$ is called *closed in* $\mathbb{C}$ if its complement $\mathbb{C} \setminus A$ is open in $\mathbb{C}$.

Remark B.5. Often, this text will refer to open or closed sets without explicitly referencing whether it is understood that these sets are open or closed in $\mathbb{C}$ or in $\overline{\mathbb{C}}$. However, context will make it clear which is meant and so no confusion should arise. It is also important to keep in mind the following two points that show that *often*, but not always, these two notions are equivalent anyway.

(1) A set $A \subset \mathbb{C}$ is open in $\overline{\mathbb{C}}$ if and only if it is open in $\mathbb{C}$.

(2) A bounded[2] subset E of $\mathbb{C}$ is closed in $\mathbb{C}$ if and only if it is closed in $\overline{\mathbb{C}}$. Note that the boundedness condition is crucial since, for example, the unbounded set $\mathbb{R}$ is closed in $\mathbb{C}$, but not closed in $\overline{\mathbb{C}}$.

The readers should take a moment to convince themselves that these two statements are indeed true.

An open set $A \subset \overline{\mathbb{C}}$ is called ***connected*** if given any two points $z, w \in A$ there exists a *polygonal line*[3] in A which connects z to w. If U is an open set and $z_0 \in U$, then the set $U(z_0)$ of all points $z \in U$ such that there is a polygonal line in U which connects z_0 to z is called the *component of U containing* z_0. Two facts regarding an open set U are as follows

(1) An open set U equals the union of components, i.e., $U = \cup_{z \in U} U(z)$.

(2) Components $U(z)$ and $U(w)$ equal each other exactly when there is a polygonal path in U which connects z to w.

The interested reader can read the details proving these facts in [**2**]; however, a less formal understanding of these concepts will suffice for this text.

A ***domain*** is a non-empty open connected set in $\overline{\mathbb{C}}$. Note that the *domain set* of a function need not be a domain in the sense just defined. A ***Jordan domain*** is any ***simply connected domain*** in $\mathbb{C}$, i.e., any domain D in $\mathbb{C}$ such that every simple closed curve in D encloses only points in D. Informally, this means that D has no "holes" in it. For example, the set $\mathbb{D}$ is a Jordan domain, but the set $\mathbb{D} \setminus \{0\}$ is not.

A ***neighborhood*** of a point $z \in \overline{\mathbb{C}}$ is any open set which contains z. A ***deleted neighborhood*** of a point $z \in \overline{\mathbb{C}}$ is any set $U \setminus \{z\}$ where U is a neighborhood of z. When a neighborhood U is of the form $\Delta_\sigma(z, \epsilon)$ or $\Delta(z, \epsilon)$ we often refer to this as an ϵ-***neighborhood*** of z.

B.3.1. Convergence in the Riemann Sphere. With the spherical metric σ on $\overline{\mathbb{C}}$ and the corresponding notion of ϵ-neighborhood, we can now define convergence of sequences in $\overline{\mathbb{C}}$ as follows.

DEFINITION B.6. A sequence of points $z_n \in \overline{\mathbb{C}}$ is said to converge to $z \in \overline{\mathbb{C}}$ when $\sigma(z_n, z) \to 0$, in which case we write $z_n \to z$.

Note that the statement $z_n \to z$ when $z \in \mathbb{C}$ (i.e., $z \neq \infty$) as given above, conforms to our standard notion of convergence in $\mathbb{C}$. The following proposition records this fact and several more which relate the standard metric in $\mathbb{C}$ to the metric σ on $\overline{\mathbb{C}}$. The reader should become comfortable with these statements, though it is not crucial to be able to prove these with the rigor of an $\epsilon - \delta$ argument.

[2] A set E is called ***bounded*** in $\mathbb{C}$ if there exists some $R > 0$ such that $|z| \leq R$ for all $z \in E$.

[3] A *polygonal line* is a union of a finite number of line segments joined end to end. In the context of $\overline{\mathbb{C}}$, we allow a line segment to be any standard line segment in the plane $\mathbb{C}$ or any arc of a great circle that passes through ∞ (as viewed as the north pole N on the sphere S).

Proposition B.7. For points z_n, z, and w in $\mathbb{C}$, we have the following:
(1) $\sigma(z_n, z) \to 0$ if and only if $|z_n - z| \to 0$,
(2) $|z| > |w|$ if and only if $\sigma(z, \infty) < \sigma(w, \infty)$,
(3) $z_n \to \infty$ if and only if $|z_n| \to +\infty$ if and only if $\sigma(z_n, \infty) \to 0$,
(4) $z_n \to 0$ if and only if $|z_n| \to 0$ if and only if $1/|z_n| \to +\infty$ if and only if $1/z_n \to \infty$.

We can use this proposition to solve the *division by zero* problem, but we must first define continuity in the context of $\overline{\mathbb{C}}$.

B.4. Continuity in the Riemann Sphere

Let $R \subset \overline{\mathbb{C}}$ and consider an *extended* complex valued function $f : R \to \overline{\mathbb{C}}$. We write $\lim_{z \to z_0} f(z) = L$ and we say the function f *approaches* L as z *approaches* z_0, if for each $\epsilon > 0$ there exists $\delta > 0$ such that $\sigma(f(z), L) < \epsilon$ whenever $z \in R$ and $0 < \sigma(z, z_0) < \delta$. We say that f is *continuous* at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$, i.e., for each $\epsilon > 0$ there exists $\delta > 0$ such that $\sigma(f(z), f(z_0)) < \epsilon$ whenever $z \in R$ and $\sigma(z, z_0) < \delta$. We say f is continuous on a set U if it is continuous at each point of U. Due to Proposition B.7, this notion of continuity conforms with our usual notion of continuity in $\mathbb{C}$ (see Section A.2).

Example B.8 (The map $1/z$ and division by zero). Our definition of continuity on $\overline{\mathbb{C}}$ allows us to solve the *division by zero* problem by defining $1/0 = \infty$. This makes sense and is natural since it is this definition that makes the function $z \mapsto 1/z$ continuous on $\overline{\mathbb{C}}$. We leave it to the reader to use Proposition B.7 to show this, and in the process show that we also have $1/\infty = 0$.

We also point out an important observation considering a rotation of the sphere $S = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbb{R}^3$. Consider rotating the sphere S about the x_1 axis, which meets S at the points $(1, 0, 0)$ and $(-1, 0, 0)$, by $180° = \pi$ radians. This is a map which takes each point of S to another point of S and so let us call this map f. Identifying S with $\overline{\mathbb{C}}$ as above, we see that $f(\infty) = 0, f(0) = \infty, f(1) = 1, f(-1) = -1$ and $f(\overline{\mathbb{R}}) = \overline{\mathbb{R}}$, where $\overline{\mathbb{R}}$ denotes the extended real line $\mathbb{R} \cup \{\infty\}$. The reader should take some time to convince themself that f is exactly the map $f(z) = 1/z$. It is for this reason that we often call $f(z) = 1/z$ a rotation of the Riemann sphere $\overline{\mathbb{C}}$.

Some texts avoid the need to speak explicitly about the spherical metric while still speaking about continuity of functions which are defined at ∞ or which take on the value ∞ at finite points $z_0 \in \mathbb{C}$. Since this approach is also valuable, we present it here. The key is to use the simple rotation $z \mapsto 1/z$ of the Riemann sphere $\overline{\mathbb{C}}$ in judicious ways to "move" ∞ to 0 so that we can then use our standard notions of continuity (and, as we shall see, analyticity) in $\mathbb{C}$ without needing to make explicit reference to the spherical metric. This is how one may prove the following result.

Theorem B.9. Let $z_0, w_0 \in \mathbb{C}$ and let $f(z)$ be a complex valued function defined in a deleted neighborhood of z_0. Then,

i) $\lim_{z \to z_0} f(z) = \infty$ if and only if $\lim_{z \to z_0} \dfrac{1}{f(z)} = 0$,

ii) $\lim_{z \to \infty} f(z) = w_0$ if and only if $\lim_{z \to 0} f\left(\dfrac{1}{z}\right) = w_0$,

iii) $\lim_{z \to \infty} f(z) = \infty$ if and only if $\lim_{z \to 0} \dfrac{1}{f(1/z)} = 0$.

By pre composing and/or post composing f with the map $z \mapsto 1/z$, the above theorem converts each statement on the left (involving ∞) to the corresponding statement on the right (which avoids a formal notion of ∞). By considering the topology of the Riemann sphere $\overline{\mathbb{C}}$ near ∞ and the continuity of the map $z \mapsto 1/z$, the details can easily be checked (and they can also be found on p. 51 of [**1**]).

The simplest and most useful class of maps which are continuous on all of $\overline{\mathbb{C}}$ is the class of rational maps.

Definition B.10. *A quotient of two polynomials is called a rational function.*

Let $f(z) = \frac{P(z)}{Q(z)}$ be a rational function in reduced form (i.e., polynomials $P(z)$ and $Q(z)$ have no common factors). Although this *formula* for f is defined only for complex values where Q is not zero, we can regard f (as a mapping into $\overline{\mathbb{C}}$) as being both defined and continuous on all of $\overline{\mathbb{C}}$. In particular, if $Q(a) = 0$, then we set $f(a) = \infty$, and we also set $f(\infty) = \lim_{z \to \infty} f(z)$. We leave it to the reader to check that this gives f the desired continuity properties; however, we do illustrate this in the following examples.

Example B.11. Let $f(z) = z^2 + 5i$ and $g(z) = \frac{3z^2-5}{z^2+2z}$. In line with the above discussion, it is then understood that $f(\infty) = \infty, g(\infty) = 3, g(-2) = \infty$, and $g(0) = \infty$. By defining these maps in this way, we see that both f and g are continuous at every point of $\overline{\mathbb{C}}$. Note that $f(\infty) = \lim_{z \to \infty} f(z) = \infty$ since (using Theorem B.9) $\lim_{z \to 0} \frac{1}{f(1/z)} = \lim_{z \to 0} \frac{1}{(1/z)^2+5i} = \lim_{z \to 0} \frac{z^2}{1+5iz^2} = 0$. We leave it to the reader to use Theorem B.9 to check the continuity of g at the points $-2, 0$, and ∞.

Example B.12. The function $f(z) = e^z$ on $\mathbb{C}$ cannot be defined at ∞ to make it continuous there. One way to see this is to note that $\lim_{x \to +\infty} e^x = \infty$, but $\lim_{x \to -\infty} e^x = 0$, which implies that the limit $\lim_{z \to \infty} e^z$ does not exist.

B.5. Analyticity in the Riemann Sphere

Just as the rotation $z \mapsto 1/z$ was used in Theorem B.9 to help us to understand the notion of *continuity* in the Riemann sphere $\overline{\mathbb{C}}$, we can also use it to extend our notion of *analyticity* on $\overline{\mathbb{C}}$.

Definition B.13 (Extended notion of analyticity in $\overline{\mathbb{C}}$).
Let $f : \text{domain}(f) \to \overline{\mathbb{C}}$, where $\text{domain}(f) \subset \overline{\mathbb{C}}$.

i) If $f(z_0) = \infty$ where $z_0 \in \mathbb{C}$, then we say that f is *analytic* at z_0 exactly when $\frac{1}{f(z)}$ is analytic at z_0.

ii) If $f(\infty) \neq \infty$, then we say that f is *analytic* at ∞ exactly when $f(\frac{1}{z})$ is analytic at 0.

iii) If $f(\infty) = \infty$, then we say that f is *analytic* at ∞ exactly when $\frac{1}{f(\frac{1}{z})}$ is analytic at 0.

One way to summarize the above definition is to say that a map is analytic in this extended sense if after "moving" each instance of ∞ to 0 (by pre composing and/or post composing with the map $z \mapsto 1/z$), we get a map which is analytic in the usual sense.

REMARK B.14. We note that a map f which is analytic at a point $z_0 \in \overline{\mathbb{C}}$ must also be continuous there (as a mapping into $\overline{\mathbb{C}}$). Hence we can easily see that the map $f(z) = e^z$ cannot be analytic at ∞ since there is no way to define f at ∞ in such a way as to make it continuous there.

EXERCISE B.15. Show that $z \mapsto \sin\frac{1}{z}$ is analytic at ∞, but that $z \mapsto \sin z$ is not.

REMARK B.16 (Poles are points of analyticity in the extended sense). When $z_0 \in \mathbb{C}$ is a pole of f of order k, then by Definition A.15, we know that we can express $f(z) = (z - z_0)^{-k}h(z)$ where h is analytic at z_0 (in the usual sense of Section A.2) and $h(z_0) \neq 0$. Thus, $\frac{1}{f(z)} = (z - z_0)^k/h(z)$ is analytic at z_0, which means we can then declare f to be analytic at z_0 by Definition B.13(i), in that extended sense. Since the word *analytic* is commonly used in this text in both the usual sense of Section A.2 and this extended sense, the reader must always be careful to use context to decide in which sense it is being used. Since the context is usually quite clear, no confusion should arise.

We note that a map f with an isolated singularity at $z_0 \in \mathbb{C}$ (see Definition A.15) is analytic in the usual sense in the case of a removable singularity and analytic in the extended sense in the case of a pole. Thus, only when the singularity is essential can the map not be regarded as analytic in either sense.

EXAMPLE B.17. Let $f(z) = \frac{1}{1+z^2}$. Since $f(\frac{1}{z}) = \frac{1}{1+(\frac{1}{z})^2} = \frac{z^2}{z^2+1}$ is analytic at 0 (and takes the value 0 at 0), we can say that f is analytic at ∞ (and $f(\infty) = 0$). Similarly, since $1/f(z) = 1 + z^2$ is analytic at $\pm i$, the map f is is analytic at the poles $\pm i$, a fact we stated more generally in Remark B.16.

EXAMPLE B.18. Let $f(z) = 3z - \frac{1}{z}$ and note that $f(\infty) = \infty$. Since $k(z) = \frac{1}{f(\frac{1}{z})} = \frac{1}{3(\frac{1}{z})-z} = \frac{z}{3-z^2}$ is analytic at zero we pronounce $f(z)$ to be analytic at ∞. Note that, we have $f'(z) = 3 + \frac{1}{z^2}$ and so $f'(\infty) = 3$. We also have $k'(z) = \frac{3+z^2}{(3-z^2)^2}$ yielding $k'(0) = 1/3 = 1/f'(\infty)$. The relationship between $k'(0)$ and $f'(\infty)$ is an important and general property which we state as follows.

LEMMA B.19. If f is analytic at ∞ with $f(\infty) = \infty$, then for $k(z) = \frac{1}{f(\frac{1}{z})}$ we have $k'(0) = 1/f'(\infty)$.

Note, when $f'(\infty) = \infty$ we use the convention that $1/\infty = 0$ (since the map $1/z$ sends ∞ to 0).

PROOF. Note that k is analytic at 0 (by the definition of f being analytic at ∞). Also note that $k(0) = 0$ (since $f(\infty) = \infty$) and so we let N denote the multiplicity of the zero of k at 0. There exists $r > 0$ such that k is analytic on $\Delta(0, r)$. Therefore, the map $f(z) = \frac{1}{k(\frac{1}{z})}$ is analytic on $\{z \in \mathbb{C} : |z| > 1/r\}$. Hence f may be represented by a Laurent series $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$ on $\{z \in \mathbb{C} : |z| > 1/r\}$. Since $\frac{1}{k(z)} = f(\frac{1}{z}) = \sum_{n=0}^{\infty} a_n z^{-n} + \sum_{n=1}^{\infty} b_n z^n$ has a pole at 0 of order N, we must have $a_N \neq 0$ and $a_n = 0$ for all $n > N$. Hence we may now factor $f(z) = z^N h(z)$ where $h(z) = \cdots + a_{N-1} z^{-1} + a_N$.

Calling $g(z) = h(1/z) = a_N + a_{N-1} z + \ldots$, we can then express $k(z) = \frac{1}{f(\frac{1}{z})} = \frac{1}{z^{-N} h(1/z)} = \frac{z^N}{g(z)}$. Thus $k'(z) = \frac{N z^{N-1} g(z) - g'(z) z^N}{[g(z)]^2}$ and so

$$k'(0) = \begin{cases} 1/a_N & \text{when } N = 1 \\ 0 & \text{when } N > 1. \end{cases}$$

Note that $h'(z) = \cdots - a_{N-1} z^{-2}$ and so as $z \to \infty$ we have

$$f'(z) = N z^{N-1} h(z) + z^N h'(z) \to \begin{cases} a_N & \text{when } N = 1 \\ \infty & \text{when } N > 1. \end{cases}$$

Thus we have $k'(0) = 1/f'(\infty)$ as promised. $\square$

Bibliography

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