CHAPTER 3

Applications to Flow Problems

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3.1. Introduction

This chapter grew out of a series of lectures prepared for an undergraduate mathematical physics course. At the time, the goal was to show the students some applications of complex function theory that connected to familiar topics from calculus and physics. The idea to look at the two dimensional flows of ideal fluids was a natural fit. Many common ideas from vector calculus are used in the development of the subject and there are numerous applications of the methods that are developed. Modeling ideal fluid flow is a standard application of *conformal mappings* and is readily found in most undergraduate complex analysis texts (see [**3**] or [**5**]). However, in preparing the notes it became apparent that there was a need for a unified treatment that included a variety of applications and extensions such as including sources and sinks in the flow, or accounting for the role played by sources or sinks at infinity. Additionally, the emphasis in many texts is on the analytic aspects of the subject and not on the geometric or visual aspects of the flows. This chapter combines all this material and more. It is self-contained and relies only on basic results from vector calculus and a standard first course in complex variables. To permit discovery and experimentation, we created the easy-to-use applet *FlowTool* that can be accessed online at http://www.jimrolf.com/explorationsInComplexVariables/chapter3.html:

- *FlowTool* plots the streamlines for the flow of an ideal fluid. It permits the user to select the number and location of sources or sinks on the boundary or in the interior of the region and then allows the strength to be dynamically varied. The applet shows steady state fluid flow in four preset regions: the entire plane, the half plane, the quadrant, and the strip.

We note that the applet is limited to certain pre-set regions and particular types of sources. To account for the wide variety of problems that may be encountered in practice, we also make use of powerful computer algebra systems such as *Mathematica*.

Vector fields arise naturally in many applications. They are used to model physical phenomena such as the velocity field of a fluid flowing in a region or the electric force field generated by a collection of charges. The geometry of the region, along with any sources or sinks in the field, determines the nature of the resulting vector field. Finding descriptions of these fields is the focus of this chapter. It is also of interest to find the integral curves of such a vector field. These are curves that are everywhere tangent to the vector field. Integral curves have different names depending on the context including *flow lines*, *stream lines*, or *lines of force.*

In this chapter we study physical situations in which complex function theory can be used to help solve the problem of finding particular instances of such vector fields and their integral curves. We focus mainly on using techniques of complex analysis to solve flow problems. To motivate our discussion and to provide a road map for where the chapter is headed, we start with an example. We omit many of the details here hoping the reader will be intrigued enough to press on with the material and learn how to solve this type of problem.

EXAMPLE 3.1. Imagine an infinitely long, very shallow, channel in which fluid flows. Because the channel is very shallow we assume this is a two dimensional flow (real flows are three dimensional, but we assume the flow is identical on all parallel planes). Orient the channel on the complex plane so that one edge runs along the real axis and the other along the horizontal line $\operatorname{Im} z = \pi$ (thus our channel has width π). Now we assume that fluid is being pumped into the channel and drained from the channel at various points along the edges. For the time being we assume that all pumps and drains operate at the same constant rate. Our goal is to describe the velocity of the fluid in the channel. We assume that the flow is in a *steady state,* meaning the velocity at a point of the domain does not change with time. For this scenario, let the fluid be pumped into the channel at equal rates at $z = 0$ and $z = 2$, and suppose fluid is drained from the channel at the same rate at the point $z = -2 + \pi i$. Note that there is more fluid being pumped into the channel than there is being drained from the channel. Figure 3.1 illustrates the situation. The curves shown are the integral curves of the underlying vector field and they represent the path that a drop of dye would follow if placed into the flow. A critical assumption in this model is that we are dealing with an *ideal fluid.* This means that the fluid is incompressible, non viscous, and there is no loss of energy due to friction between the walls of the channel and the fluid. Such fluids do not really exist, but they frequently provide good approximations to real physical situations and they have certain mathematical properties that lend themselves to analysis and modeling.

The plot in figure3.1 was generated using the computer algebra system *Mathematica.* Software packages such as *Mathematica, Maple,* and *Matlab* are very powerful and can be useful in generating complicated graphics. However, they may not be easy to use or to adapt to a particular problem. In order to facilitate your understanding of this chapter we have provided the *FlowTool* applet. This applet allows you to experiment with complicated flows for a fixed number of domains. We will make use of this applet throughout the chapter and in numerous exercises. We also provide the basic syntax for generating these types of plots in *Mathematica.*

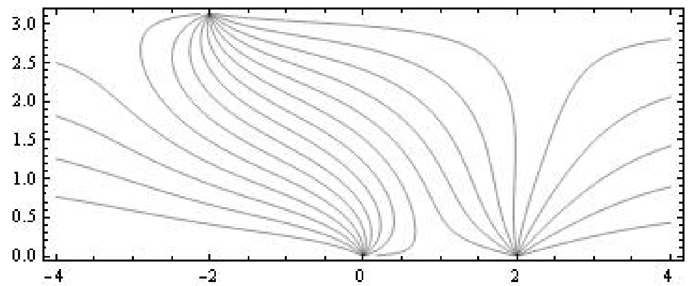

FIGURE 3.1. Flow Lines for an Ideal Fluid in a Channel with Sources and Sinks

Before moving on you should open the *FlowTool* applet and follow the instructions below to generate your own version of Figure 3.1. Open the applet and select the strip domain from the drop down menu. Uncheck the box marked Equipotential lines (we only wish to view the flow lines in this example). Use the mouse to place a source at $z = 0$, $z = 2$, and $z = -2 + \pi i$ by clicking once near each of the three locations. Notice that a dialog box with a slider opens in the right hand panel each time a source is placed on the boundary of the channel. The sliders are used to change the strength of a source. If the slider is moved to the left, the strength will eventually become negative, signifying that the source has changed to a sink. Move the slider to create a sink of strength -1 at the location $z = -2 + \pi i$. The other two locations are sources of strength 1. Figure 3.2 shows a graph similar to what you should see on the applet. You should feel free to experiment further by adding or deleting sources and using the sliders to vary the relative strengths. Observe how the flow lines run parallel to the edges of the channel, indicating that the boundary of the region acts like a frictionless barrier to the flow.

There is a lot of interesting mathematics behind the plot of the flow lines in figures 3.1 and 3.2. This chapter follows a path that develops all the key ideas needed to solve flow problems in various regions with combinations of sources and sinks along the boundary. In addition, later sections in the chapter extend the ideas and methods to more complicated settings.

Section 3.2 starts with a review of basic ideas from vector calculus including the important notions of the divergence and curl of a vector field and the definition of a harmonic function. This material should be familiar to most students.

Section 3.3 provides a review of the main results needed from a standard course in complex variables. This section also makes the critical connection between planar vector fields and complex functions. We also introduce the Polya field and give a mathematical description of sources and sinks.

Section 3.4 introduces the complex potential function which is the main tool for constructing ideal fluid flows. We make several key connections between the complex potential function and the underlying vector field. This material leads directly to the

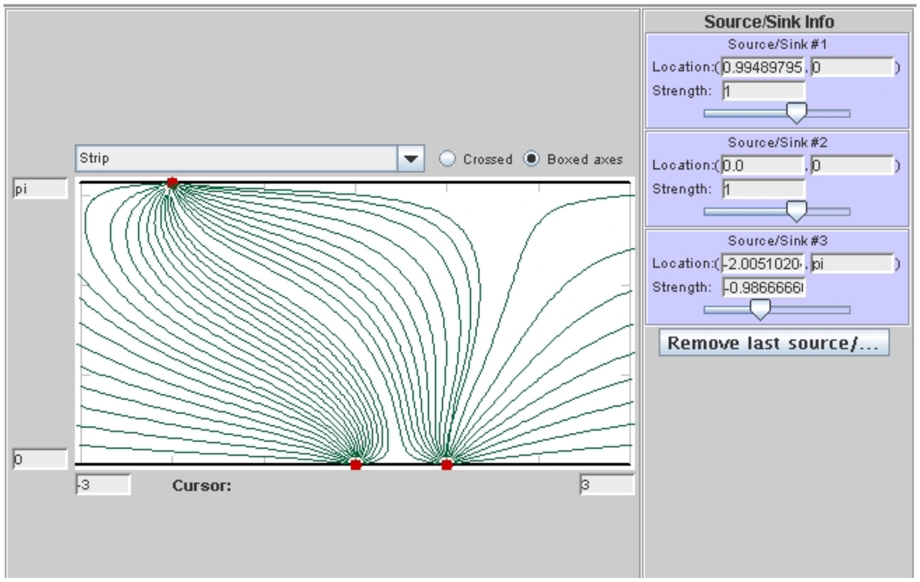

FIGURE 3.2. Flow Lines in a Channel with Sources and Sinks generated with FlowTool Applet.

construction of uniform flows in various regions in Section 3.5. We then continue the development in Section 3.6 by allowing sources and sinks along the boundary of the region. We then take a careful look at the uniform flow in a channel in Section 3.7. This development sheds light on the important role played by sources or sinks at infinity. Section 3.8 puts all the previous material together by discussing flows in any region with various combinations of sources or sinks along the boundary.

The final five sections focus on various applications or extensions of the material. These topics include interval sources, sources or sinks in the interior of regions, flows inside disks, dipoles, and steady state temperature problems.

3.2. Background and Fundamental Results

Several fundamental concepts from multivariable calculus are required for this material. We briefly review the main ideas here. The reader wishing to obtain more background regarding vector fields and their associated operations should consult any standard calculus text such as [**1**].

We represent a 2-dimensional or planar vector field in Cartesian coordinates using two real-valued functions of position: $\vec{F}(x, y) =< P(x, y), Q(x, y) >$. Simple examples include constant vector fields such as $\vec{F}(x, y) =< 3, 4 >$ and the field tangent to concentric circles about the origin given by $\vec{G}(x, y) =< -y, x >$. Other examples include slope fields for first order differential equations or the magnetic field in a plane perpendicular to a wire with a current flowing through it. Vector fields are represented graphically in the plane by drawing arrows indicating the direction of the field at selected points. The magnitude of the field at a point is given by

$$|\vec{F}| = \sqrt{P(x,\, y)^2 + Q(x,\, y)^2}$$

and the direction θ satisfies

$$\tan\theta = \frac{Q(x,\, y)}{P(x,\, y)}$$

for $P(x,\, y) \neq 0$. If $Q(x,\, y) \neq 0$ and $P(x,\, y) = 0$, then the direction is $\pm\frac{\pi}{2}$ at $(x,\, y)$, the choice being determined by the sign of $Q(x,\, y)$. The vector $< 0,\, 0 >$ has no defined direction. It is usually impossible to draw vectors with their true magnitude, so graphical representations frequently scale the length of the vectors down to a more manageable size. Figure 3.3 shows a constant vector field, while Figure 3.4 shows the magnetic field generated by a current running perpendicular to the plane through the origin. We also show an integral curve of the field.

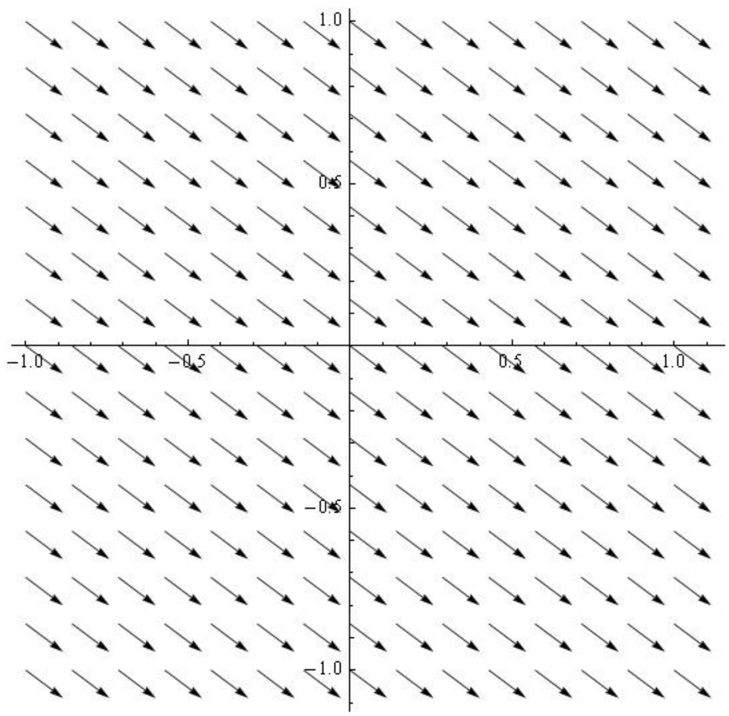

FIGURE 3.3. The constant vector field given by $\vec{F}(x,\, y) =< 4, -3 >$.

The *curl* and *divergence* of $\vec{F}$ are defined respectively in terms of the vector differential operator $\nabla =< \partial_x,\, \partial_y >$ as follows:

$$\nabla \times \vec{F} = (Q_x - P_y)\hat{k} \text{ (Curl)} \tag{21}$$

$$\nabla \cdot \vec{F} = P_x + Q_y \text{ (Divergence)} \tag{22}$$

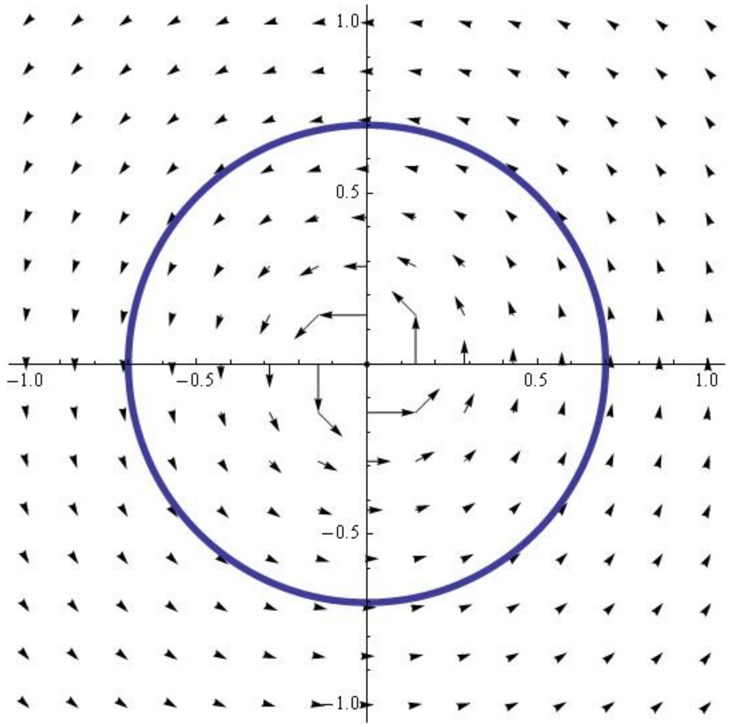

FIGURE 3.4. Magnetic field generated by a current in a wire. The circle represents an integral curve of the field.

The curl is really defined on 3-dimensional vector fields and yields another vector field orthogonal to the original field. However, in the case of 2-dimensional planar fields we simply assume the $\hat{k}$-component of the vector field $\vec{F}$ is zero, implying that the curl will always point in the $\hat{k}$-direction. Thus, it is sufficient to simply compute the scalar component of $\nabla \times \vec{F}$, namely $Q_x - P_y$.

We say that a vector field is *irrotational* at a point if the curl is zero. Similarly, we say the vector field is *incompressible* at a point if the divergence is zero. The standard physical description of a fluid with zero curl is that an infinitesimally small paddle wheel placed horizontally into the flow would not rotate. That is the flow contains no vortices. The physical description of the divergence being zero is the idea of the fluid being incompressible. One consequence of this is that the amount of fluid entering some region must be equal to the amount of fluid leaving the same region.

In this chapter we study vector fields that are *both* irrotational and incompressible throughout some region. Whereas this may seem restrictive, it actually encompasses a wide range of physically meaningful phenomena such as the electric field in a region free from charges or the velocity field of an ideal fluid flowing in a region. In addition, the gravity force field generated by a collection of masses also falls in this category.

The requirement that both the curl and the divergence be zero imposes conditions on the component functions of the vector field $\vec{F}(x,\, y) =< P(x,\, y),\, Q(x,\, y) >$. From Equations (21) and (22) we obtain a pair of partial differential equations relating the components of $\vec{F}$:

$$
\begin{aligned}
P_x &= -Q_y \\
P_y &= Q_x
\end{aligned}
\tag{23}
$$

Note the similarity of these equations to the Cauchy-Riemann equations satisfied by analytic functions. Indeed, if $f(z) = u(x,\, y) + iv(x,\, y)$ is an anlytic function, then the real and imaginary parts satisfy the Cauchy-Riemann equations:

$$
\begin{aligned}
u_x &= v_y \\
u_y &= -v_x
\end{aligned}
\tag{24}
$$

EXERCISE 3.2. Determine if the following vector fields are incompressible and/or irrotational:

(1) $\vec{F}(x, y) =< ax + by,\, cx + dy >$, where a, b, c, and d are real
(2) $\vec{G}(x, y) =< e^x \cos y,\, e^x \sin y >$

Try it out!

Recall that given a vector field $\vec{F}(x,\, y)$ and a path C,, the line integral of $\vec{F}$ along C is denoted by $\int_C \vec{F} \cdot d\vec{R}$ and measures the work done by the field on a particle traversing the path C. The line integral is computed by finding a parameterization of C, given by $r(t) = (x(t),\, y(t))$, where $a \le t \le b$, and computing $\int_a^b \vec{F}(r(t)) \cdot r'(t)\, dt$.

A vector field $\vec{F}$ is said to be *conservative* in a simply connected domain D if the line integral $\int_C \vec{F} \cdot d\vec{R}$ between any two fixed points in D is independent of the path C chosen for the integration (provided the path lies entirely in D). There are numerous equivalent conditions that guarantee a field is conservative. See [**1**] for a complete discussion.

THEOREM 3.3. Let $\vec{F}$ be a vector field with component functions that are continuous and have continuous first order partial derivativesthroughout a simply connected region D. The following statements are equivalent:

(1) $\vec{F}$ is conservative.
(2) There exists a differentiable *potential* function ϕ such that $\nabla\phi = \vec{F}$, where $\nabla\phi =< \phi_x,\, \phi_y >$.
(3) $\int_C \vec{F} \cdot d\vec{R} = 0$ for every closed loop C in D.
(4) $\nabla \times \vec{F} = 0$.

We make extensive use of the result that a curl-free vector field has a potential function. The condition $\nabla\phi = \vec{F}$ implies the system of partial differential equations

$$\begin{aligned} \phi_x &= P(x,\, y) \\ \phi_y &= Q(x,\, y) \end{aligned} \tag{25}$$

be satisfied. These equations allow us to construct potential functions through partial integration. The function ϕ is unique up to an additive constant (why?) and is called the *real potential function* of $\vec{F}$.

Another key definition that plays an important role in our work is that of a *harmonic function.* These functions are solutions of Laplace's equation. Harmonic functions play a critical role in much of applied mathematics. They arise frequently as steady-state solutions to various physical problems. Harmonic functions and their properties are closely tied to the theory of analytic functions in complex analysis. In solving flow problems, harmonic functions play a central role.

DEFINITION 3.4. Let $u : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ have continuous 2nd order partial derivatives. If u satisfies Laplace's Equation $\Delta u = \nabla \cdot \nabla u = u_{xx} + u_{yy} = 0$, then u is said to be a harmonic function in D.

The following exercises explore the mechanics of working with vector fields, potential functions, and harmonic functions. In particular, Exercise 3.6 shows that the potential function for an irrotational and incompressible field must be harmonic.

EXERCISE 3.5. Let $\vec{F}(x, y) =< x^3 - 3xy^2,\, y^3 - 3x^2y >$. Compute both the curl and divergence of $\vec{F}$. If $\vec{F}$ is conservative, then find a potential function for it. If $\vec{F}$ is also incompressible, then show that the potential function is harmonic. (Optional) Use a program such as Mathematica or Matlab (or some other utility) to graph both $\vec{F}$ and several level curves of the potential function. What key geometrical observation connects the direction of $\vec{F}$ with the tangents to the level curves of the potential function? ***Try it out!***

EXERCISE 3.6. Suppose that $\vec{F}$ is both irrotational and incompressible. Let ϕ be a real potential function of $\vec{F}$. Show that ϕ is a *harmonic function*. ***Try it out!***

The result of Exercise 3.6 is particularly important. Note that it is the irrotational feature of the vector field that implies the existence of the potential function and the incompressibility then implies the potential function is harmonic.

EXERCISE 3.7. Consider an attracting force at the origin of the xy-plane whose magnitude at a point is inversely proportional to the square of the distance from the origin to the point. Determine a formula for the vector field representing this force field and determine if the field is both irrotational and incompressible in any region not containing the origin. If possible, find a potential function for the vector field. Finally, change the field by assuming the force of attraction at a point is inversely proportional to the distance to the point (as opposed to the square of the distance). Now compute the curl and divergence of this field. How do your answers compare? ***Try it out!***

3.3. Complex Functions and Vector Fields

In this section we make several basic connections between vector fields and complex functions. A vector field being both irrotational and incompressible is closely related to the concept of an analytic function.

Any complex function of the complex variable $z = x + i\,y$ can be expressed in terms of its real and imaginary parts as $f(z) = u(x, y) + i\,v(x, y)$, where u and v are real-valued functions of x and y. This is accomplished by setting $z = x + i\,y$ and separating into real and imaginary parts. Recall that the function f is said to be *analytic* at $z = z_0$ if the complex derivative $f'(z)$ exists for every z in a neighborhood of z_0. An immediate and far-reaching consequence of this definition is that the functions u and v must satisfy the Cauchy-Riemann equations:

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned} \tag{26}$$

We recall a basic result about analytic functions:

THEOREM 3.8. Let $f(z) = u(x, y) + iv(x, y)$ be analytic in a domain D, then the functions u and v are both harmonic in D.

This result is an easy consequence of the Cauchy-Riemann equations and the reader is strongly encouraged to prove the result in the following exercise.

EXERCISE 3.9. Prove Theorem 3.8. ***Try it out!***

An equally important result is that every harmonic function on a simply connected domain is the real part of an analytic function. For a complete discussion, see [**5**].

THEOREM 3.10. Let $u(x, y)$ be a harmonic function defined on a simply connected domain D. Then, there exists another harmonic function $v(x, y)$ defined on D called the *harmonic conjugate* of u, such that $f(z) = u(x, y) + iv(x, y)$ is analytic on D. The harmonic conjugate is not unique.

EXERCISE 3.11. Find a harmonic conjugate for $u(x,\, y) = 3x^2 - 2y - 3y^2$. ***Try it out!***

EXERCISE 3.12. Let $u(x, y) = \frac{1}{2}\mathrm{Log}(x^2 + y^2)$. Show that u is harmonic on the punctured plane. Next, omit any infinite ray emenating from the origin and construct a harmonic conjugate for u on the plane minus the ray. ***Try it out!***

Next we establish a correspondence between the set of complex functions and the set of planar vector fields. Indeed, since both complex functions and planar vector fields can be represented by a pair of real-valued functions, any complex function may be thought of as a vector field and visa-versa. We note the following correspondence:

$$V(z) = P(x,y) + iQ(x,y) \iff \vec{F}(x,y) = < P(x,y),\, Q(x,y) > \tag{27}$$

Initially the correspondence is only a renaming of one object into another with no obvious value. However, looking at complex functions as vector fields sometimes yields insights that are not apparent in the more traditional view of complex functions as mapping regions from one copy of the complex plane to another copy. When plotting the vector field of a complex function, the magnitude of the vector field at z is $|V(z)|$ and the direction is $\arg(V(z))$. For example, Figure 3.5 shows the vector field corresponding to the complex function $V(z) = z^2$

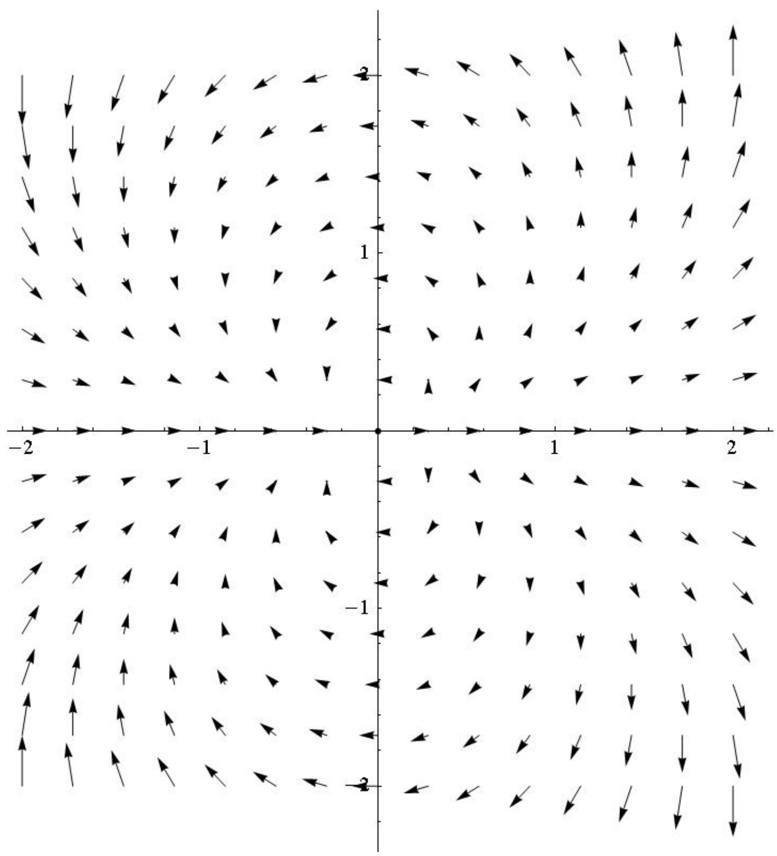

FIGURE 3.5. Vector field representation of the function $V(z) = z^2$

For the remainder of this chapter we will use the notation $V(z)$ to refer to both complex functions and their corresponding vector field representation.

The correspondence given by (27), though particularly simple, is not the most useful vector field representation of a complex function. For reasons that will become clear, we instead associate a complex function with its *Polya vector field*.

DEFINITION 3.13. Let $V(z) = u(x, y) + i\,v(x, y)$ be a complex function. The vector field given by the conjugate $\overline{V(z)} = u(x, y) - i\,v(x, y)$ is called the *Polya vector field* of $V(z)$.

It is important to note then when plotting $\overline{V(z)}$, that we still attach the arrow to the point z, not $\bar{z}$.

The Polya vector field of $V(z) = z^2$ is $\overline{V(z)} = \bar{z}^2$ and is shown in Figure 3.6. Contrast this field with the one shown in Figure 3.5. We will see the value of the Polya field representation when we look at the vector field corresponding to an analytic function.

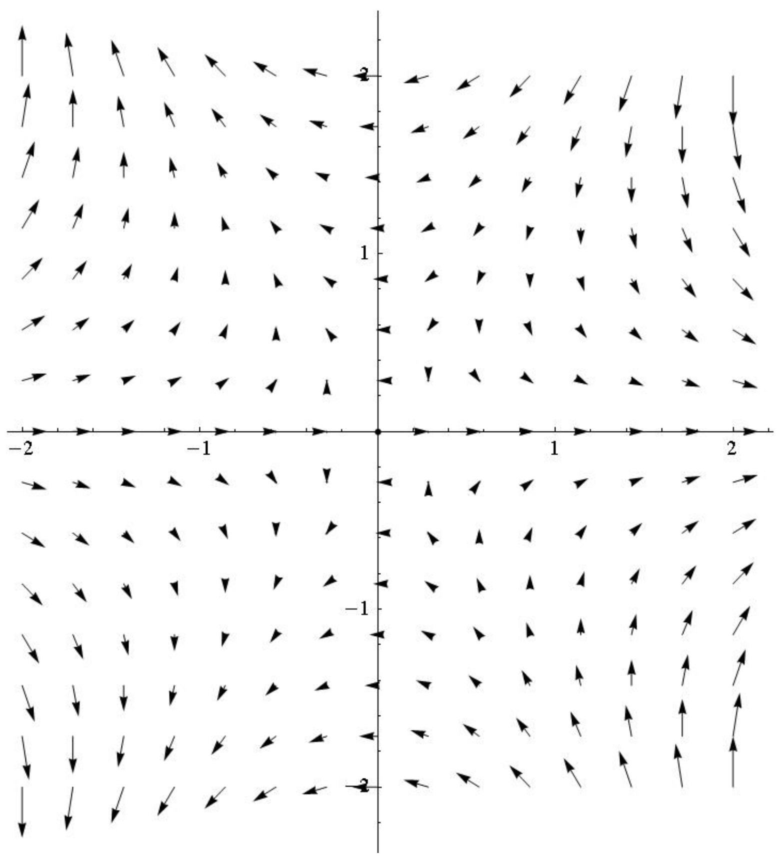

FIGURE 3.6. The irrotational, incompressible Polya field $\overline{V(z)} = \bar{z}^2$

The following theorem shows that if a complex function is analytic, then its Polya field is both irrotational and incompressible. The proof is a straightforward calculation involving the Cauchy-Riemann equations and is left as an exercise.

THEOREM 3.14. Let $V(z)$ be a complex function which is twice differentiable on a domain D. $V(z)$ is analytic on D if and only if its Polya vector field $\overline{V(z)}$ is irrotational and incompressible on D.

EXERCISE 3.15. Prove Theorem 3.14. ***Try it out!***

The previous Theorem 3.14 establishes a bijection between the set of irrotational, incompressible planar vector fields and the set of analytic functions. It is precisely this connection that allows us to use the techniques of complex analysis to solve flow problems.

EXERCISE 3.16. Find the component functions of each irrotational and incompressible vector field corresponding to the following analytic functions. Verify directly that the resulting vector fields have both curl and divergence equal to zero.

(1) $V(z) = z^2$
(2) $V(z) = e^z$
(3) $V(z) = \frac{1}{z}$

Try it out!

EXERCISE 3.17. Verify that the following vector field is irrotational and incompressible and then find the corresponding analytic function.

$$\overline{V(z)} = (x^2 - y^2 - 2x) + i\,(-2xy + 2y).$$

Try it out!

The third example in Exercise 3.16 deserves some additional comments. The function $V(z) = 1/z$ is not defined at $z = 0$, but is analytic at every point in any neighborhood of $z = 0$. Such a point is called a *singular point* or a *singularity* of the function. The corresponding Polya vector field $\overline{V(z)}$ is also undefined at $z = 0$. However, graphing the vector field reveals that the field lines emanate radially from the origin; see Figure 3.7. We call such a point a *source.* Conversely, if the field lines went towards the origin, we would refer to the origin as a *sink*. This leads to the following definition.

DEFINITION 3.18. Let $V(z)$ be an analytic function except at $z = 0$. If $V(z)$ has a pole of order 1 at $z = 0$, then zero is either a source or sink of the corresponding Polya field $\overline{V(z)}$. The sign of the coefficient of the z^{-1}-term of the Laurent expansion of $V(z)$ about $z = 0$ determines whether the point is a source or a sink.

Sources and sinks have physical interpretations in terms of some quantity being created or destroyed. In the case of flow problems we interpret a source as a location where fluid is pumped into the region, whereas a sink acts like a drain, removing fluid from the region. In the case of an electric field, sources and sinks correspond to positive and negative charges respectively. For gravitational fields, point masses correspond to sinks of various strengths (depending on the mass). Allowing vector fields to have sources and sinks greatly extends the examples to which we may apply our methods.

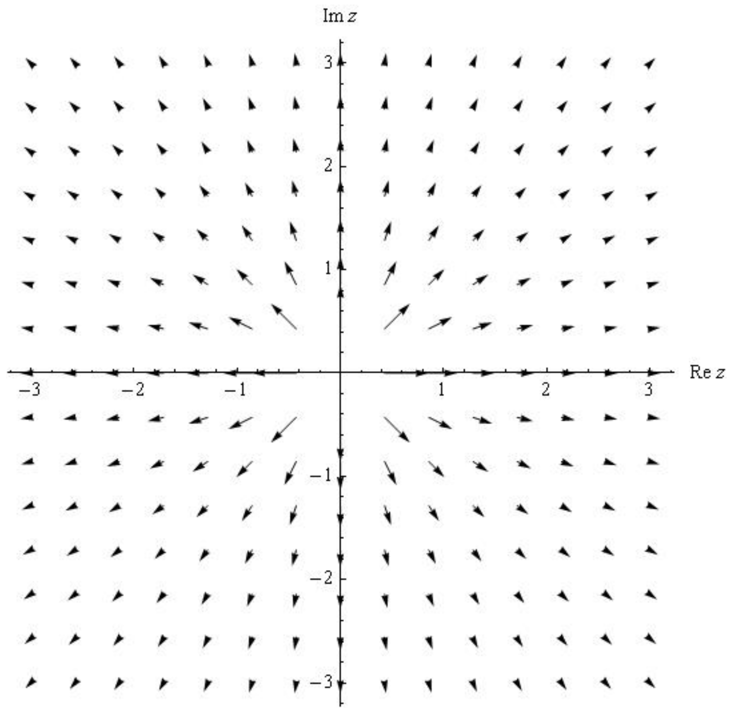

FIGURE 3.7. Polya vector field of the analytic function $f(z) = \frac{1}{z}$

EXERCISE 3.19. Let $V(z) = 1/z$. Show that the Polya field $\overline{V(z)}$ corresponds to the familiar inverse square law from physics. That is, show that the vector field emanates radially from the origin and that the magnitude of the field at a point in the plane is inversely proportional to the distance from the point to the origin. ***Try it out!***

If we desire to have a vector field with a source at some point $z = z_1$ rather than $z = 0$ we only need to translate the corresponding analytic function to obtain $V(z) = 1/(z - z_1)$. A sink is obtained by changing the sign of the function. The relative strength of a source or sink is changed by multiplying $V(z)$ by a real scalar. Finally, to account for more than one source or sink we use the *Principle of Superposition.* This principle states that if several components act to generate a vector field, then the field is obtained as the sum of the fields generated from each component separately. This is nothing more than the linearity of vector addition, but the consequences are far-reaching: complicated systems may be analyzed by studying the simpler components from which they are generated. For example, Figure 3.8 shows the field generated by a sources at $z = 0$ and $z = 1$ and sink of twice the strength of either source located at $z = 1 + 2i$. The analytic function corresponding to this field is given by

$$V(z) = \frac{1}{z} + \frac{1}{z-1} - \frac{2}{z-(1+2i)}. \tag{28}$$

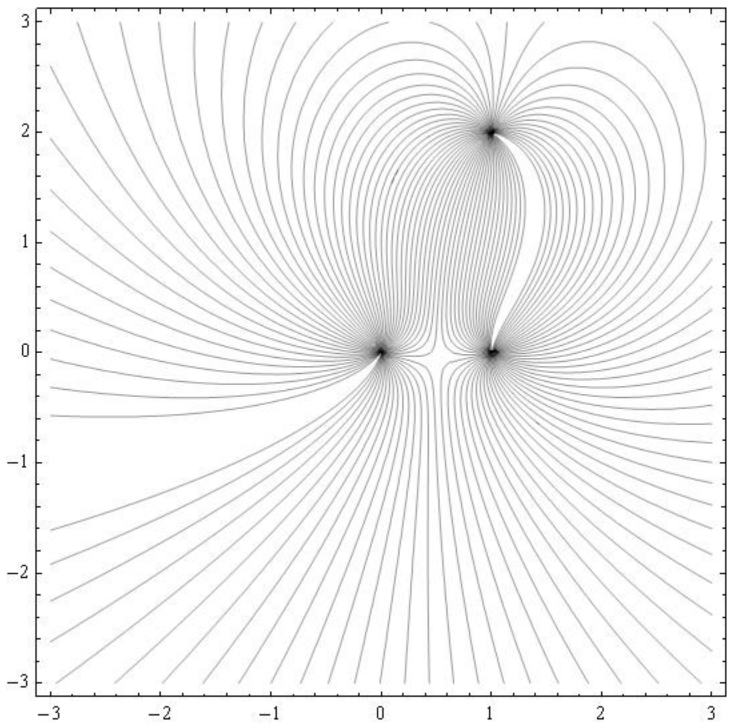

FIGURE 3.8. Vector field with two point sources and one sink. Integral curves are shown for clarity rather than arrows.

From a qualitative standpoint only the relative strengths of any sources or sinks matters in the analysis of the situation. Nevertheless, we compute the actual strength of a source or sink for completeness. We define the strength of a source to be the flux of the vector field over any simple closed loop C that encloses the source, but no other singularities of the function.

DEFINITION 3.20. Let $\vec{F}(x, y) =< P(x, y), Q(x, y) >$ be an irrotational, incompressible vector field with a source at $z = z_0$. The strength of the source is defined to be

$$\int_C P\,dy - Q\,dx, \tag{29}$$

where C is a loop traversed in a counter clockwise direction containing z_0 and no other singularities of the field. If the strength is negative, then the source is actually a sink.

Equation (29) is the familiar formula for computing the flux of a vector field.

There is an interesting relationship between the flux and circulation of a vector field and the complex integral of the associated Polya field. Indeed, the strength of the source can be determined directly from the complex integral $\int_C \overline{V(z)}\,dz$ as shown in the following important exercise.

Exercise 3.21. Let $V(z) = u(x, y) + i\,v(x, y)$ be a vector field Let C be a simple loop not passing through any singular points of the field. Expand the integral $\int_C \overline{V(z)}\,dz$ in real and imaginary parts. (Hint: let $dz = dx + i\,dy$.) Show that

$$\int_C \overline{V(z)}\,dz = \int_C u\,dx + v\,dy + i\int_C u\,dy - v\,dx.$$

This computation shows that the flux of $V(z)$ outward across C is obtained from the imaginary part of the integral $\int_C \overline{V(z)}\,dz$. Note as a consequence that the circulation of $V(z)$ is given by the real part of the integral. ***Try it out!***

The above exercise gives a useful characterization and interpretation of complex integration in terms of the familiar concepts of work and flux. An immediate consequence is that if $\overline{V(z)}$ is irrotational and incompressible both inside and on C, then both the work along C and the flux across C must both be zero. This implies that the function $V(z)$ is analytic (as was already established). An exceptional discussion of Polya vector fields and their relationship to complex integration can be found in Needham [**2**].

In particular, if $V(z)$ is analytic everywhere except for a finite number of simple poles located at $z_i, i = 1, 2, \ldots, n$, then the strength of the source or sink of $\overline{V(z)}$ located at z_k is given by the imaginary part of $\int_C V(z)\,dz$, where C is a simple closed loop traversed in the counter clockwise direction such that z_k is contained in the interior of C and none of the other z_i are inside C.

Let C be a simple closed curve containing z_k, but no other z_i for $i \neq k$. The function $V(z)$ can be represented in the form

$$\frac{g(z)}{z - z_k},$$

where $g(z)$ is analytic inside and on C. By Cauchy's Integral Formula, $\int_C V(z)\,dz = 2\pi i g(z_k)$.

Exercise 3.22. In constructing vector fields with sources or sinks, the most typical case is where $V(z)$ has the form

$$V(z) = \sum_{j=1}^{n} \frac{a_j}{z - z_j},$$

where $a_j \in \mathbb{R}$. Show that for the corresponding Polya field that the strength of the source or sink located at $z = z_k$ is simply $2\pi a_k$. ***Try it out!***

Exercise 3.23. Let $a \in \mathbb{R}$. Show that the vector field $\overline{V(z)} = a/(\bar{z} - \overline{z_0})$ has a source of strength $2\pi a$ at $z = z_0$ by directly computing the imaginary part of $\int_C V(z)\,dz$, where C is a circle centered at z_0. Do this by using a parameterization of C as opposed to Cauchy's Integral Formula. ***Try it out!***

Exercise 3.24. Consider the Polya field of the function

$$V(z) = \frac{\sin z}{z^2}.$$

This vector field has a simple pole at the origin. What is the strength of the source? ***Try it out!***

3.4. Complex Potential Functions

At this point, we have established a one-to-one correspondence via complex conjugation between irrotational, incompressible vector fields in the plane with a finite collections of point sources and sinks and complex functions which are analytic except at finitely many points where they have poles of order one.

Having established this correspondence we now proceed to construct the *complex potential function* of the underlying vector field. Let $V(z)$ be analytic on a simply connected domain so that $\overline{V(z)}$ is irrotational and incompressible, hence $\overline{V(z)}$ has a potential function ϕ that is harmonic. Thus, by Theorem 3.10 we know there exists another harmonic function ψ such that the complex function $\Omega(z) = \phi(x, y) + i\,\psi(x, y)$ is analytic. Note that ψ is only determined up to an additive constant. The function Ω is called a complex potential function of $\overline{V(z)}$. The function Ω tells us a great deal about the underlying vector field.

Definition 3.25. Given an irrotational, incompressible vector field $\overline{V(z)}$ on a simply connected domain, a *complex potential function* of $\overline{V(z)}$ is the *analytic* function $\Omega(z) = \phi(x, y) + i\,\psi(x, y)$, where ϕ is the real potential function of $\overline{V(z)}$ and ψ is a harmonic conjugate of ϕ.

It should be noted that a globally defined complex potential function may not exist. For example, $\overline{V(z)} = 1/\bar{z}$ is irrotational and incompressible on the punctured plane, but a complex potential for this field involves choosing a branch of the complex logarithm function which is not analytic on any neighborhood containing the origin.

As we already know, the level curves of ϕ (the real potential function) are orthogonal to the direction of $\overline{V(z)}$. Thus these curves form the *equipotential* lines of the field. In the case of a velocity field of an ideal fluid, these curves represent points where the velocity is constant. Whereas in the case of an electric field, the level curves of ϕ represent curves of constant electrostatic potential. So, what do the level curves of ψ represent? The answer is found in the following standard result from complex variables. We restate the result here.

Theorem 3.26. Let $\Omega(z) = \phi(x, y) + i\,\psi(x, y)$ be analytic at $z_0 = x_0 + i\,y_0$ and suppose that $\Omega'(z_0) \neq 0$. Then the tangents to the level curves of ϕ and ψ are orthogonal at the point $(x_0,\, y_0)$.

For a proof see Zill and Shanahan [**3**].

Theorem 3.26 implies that the level curves of ψ are parallel to the underlying vector field $\overline{V(z)}$. Thus, these level curves are the integral curves of the field. They are, in fact, exactly the curves that were sketched by the *FlowTool* applet in the opening example to this chapter. The function ψ is often referred to as the *stream function* in the case of a flow problem. The level curves of ψ are called the *stream lines.*

The vector field can be obtained from the complex potential function according to the following exercise.

EXERCISE 3.27. Let $\overline{V(z)}$ be an irrotational and incompressible vector field with analytic complex potential function $\Omega(z)$. Show that $\Omega'(z) = V(z)$. (Hint: Write $\Omega(z)$ in terms of its real and imaginary parts and differentiate using the Cauchy-Riemann equations.) ***Try it out!***

An equivalent statement to the preceding exercise is that

$$\Omega(z) = \int V(z)\,dz. \tag{30}$$

This formulation is useful in the following exercise:

EXERCISE 3.28. Let $\overline{V(z)} = 3\bar{z}^2 - 4\bar{z}$ be an irrotational, incompressible vector field.

(1) Determine a complex potential function on $\mathbb{C}$ using Equation (30).
(2) Find the real and imaginary parts of the complex potential function to easily find the real potential function and the stream function.

Try it out!

We end this section with an exercise that illustrates a mapping property of the complex potential function.

EXERCISE 3.29. Let Ω be the complex potential for an irrotational, incompressible vector field. Show that when viewed as a mapping of the z-plane to the w-plane, Ω maps equipotential curves to vertical lines and flow lines to horizontal lines. ***Try it out!***

3.5. Uniform Flows in the Plane and other Regions

We now begin the process of building up examples that allow us to solve a wide variety of problems in different regions of the plane. We start by considering very simple flows in the entire plane. Consider the function $\Omega(z) = z$. Thinking of Ω as a complex potential function, the natural question is to determine the underlying vector field. The following exercise asks you to determine this field.

EXERCISE 3.30. Show that $\Omega(z) = z$ is the complex potential function for a uniform flow to the right. Be sure to explicitly compute the vector field $V(z)$. See Exercise 3.27. Also show directly that the level curves of the imaginary part of Ω are stream lines for the flow. ***Try it out!***

EXERCISE 3.31. How does the answer to Exercise 3.30 change if the complex potential function is $\Omega(z) = (2+3i)z$? Show that the stream lines are given by the family of linear equations $3x + 2y = c$. See Figure 3.9. ***Try it out!***

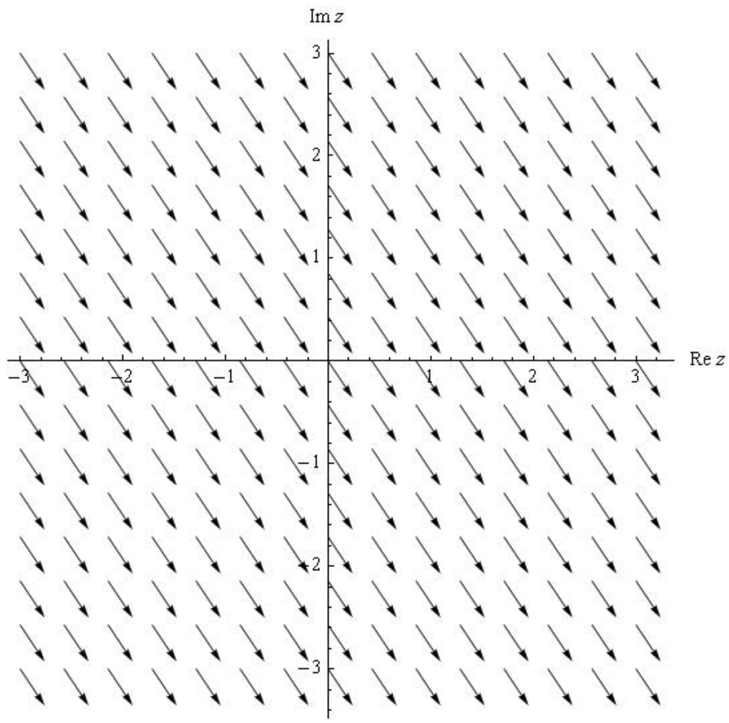

FIGURE 3.9. Uniform Flow with complex potential $\Omega(z) = (2 + 3i)z$

Our goal is to solve flow problems in various regions such as sectors, strips, and disks. However, to this point we have only studied flows in the entire complex plane. The connection lies in the theory of conformal mappings. The subject of conformal mappings is central to any course in complex analysis. In brief, a conformal mapping is a one-to-one complex function mapping some region in the z-plane to some region in the w-plane in such a way that angles and orientation are preserved. A key result is that an analytic function is conformal at all points where its derivative does not equal zero. The critical property of conformal mappings for our development is the following theorem:

THEOREM 3.32. Let $\Omega(w)$ be analytic on a domain $D' \in \mathbb{C}$, and let $f : D \to D'$ be conformal for some domain $D \in \mathbb{C}$. Then the composition $\widetilde{\Omega}(z) = (\Omega \circ f)(z)$ is analytic on the domain D.

For a proof see Zill and Shanahan [**3**].

The significance of Theorem 3.32 is that it allows a given problem to be translated to a simpler domain, solved, and then translated back to the original domain. Unfortunately, it is usually impossible to conformally map a given region to the entire complex plane. Instead, we make use of a deep and powerful theorem from complex analysis.

Theorem 3.33 (Riemann Mapping Theorem). Every simply connected domain in the complex plane, with the exception of the entire plane, can be conformally mapped to the upper half plane, $\mathbb{H} = \{z \mid \operatorname{Im} z > 0\}$.

The Riemann Mapping Theorem gives no indication of how to find the needed conformal mapping, just that one exists. Nevertheless, there are extensive tables of conformal mappings that give specific instances of the needed mappings for a large collection regions. A good table of conformal mappings can be found in [**3**]. Despite being only an existence result, Theorem 3.33 indicates that it is valuable to know how to solve problems on the upper half plane $\mathbb{H} = \{z \mid \operatorname{Im} z > 0\}$ in $\mathbb{C}$. Indeed, we will translate many different problems to equivalent problems in $\mathbb{H}$ in order to solve them.

Suppose an ideal fluid is flowing from left to right in $\mathbb{H}$. The real axis in this case acts as a boundary for the flow. That is the real axis acts as a streamline for the flow. It is clear that a constant vector field such as $\overline{V(z)} = 1$ is such a flow. It is easy to see the complex potential of this field is $\Omega(z) = z$. Thus, the identity function may be regarded as the complex potential for a uniform flow to the right in $\mathbb{H}$. Our goal is to use this fact, combined with the theory of conformal mappings to find the complex potential for flows in other regions and in regions that have sources and sinks.

Our first example deals with a uniform flow in a quadrant, often called *flow around a corner*.

Example 3.34. Imagine if a vertical barrier were inserted along the imaginary axis into the uniform flow in $\mathbb{H}$ described above. The result is likely to look like the flow shown in Figure 3.10. The goal is to find the complex potential function for this flow. The key is Theorem 3.32. We seek a conformal mapping that maps the region shown in Figure 3.10 to $\mathbb{H}$ with the property that the boundaries of the region are mapped to the real axis. This last part is important because the boundaries of the region are always streamlines for the flow (that is the flow is parallel to the boundaries and there is no friction). If we let $h(z) = -z^2$ then this function will map the 2nd quadrant to $\mathbb{H}$. Observe that the negative real axis is mapped to the negative real axis and the imaginary axis is mapped to the positive real axis. We know that the complex potential for the uniform flow in $\mathbb{H}$ is given by $\Omega(z) = z$, so we set $\widetilde{\Omega}(z) = \Omega(h(z)) = h(z) = -z^2$. Note that $\widetilde{\Omega}(z) = -x^2 + y^2 - 2xyi = -z^2$ and that the underlying vector field is $\overline{\widetilde{\Omega}'(z)} = -2\bar{z}$. Plotting the streamlines gives Figure 3.10. The streamlines are given by the level curves of the imaginary part of $\widetilde{\Omega}$, namely $-2xy = c$ for different values of the constant c.

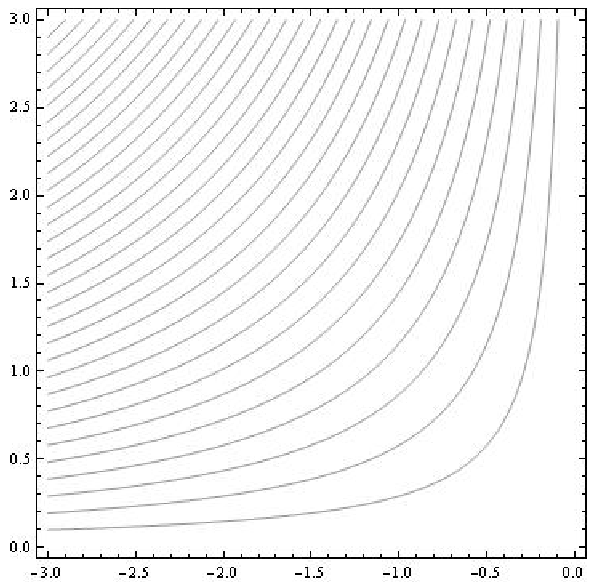

FIGURE 3.10. Uniform Flow around a Corner

EXERCISE 3.35. Let R be the region given in polar coordinates by

$$R = \{(r, \theta) \,|\, r \geq 0 \text{ and } 0 \leq \theta \leq \frac{\pi}{4}\}.$$

Find the complex potential function for a uniform (no sources or sinks) irrotational, incompressible flow in R. ***Try it out!***

The discussion above along with Exercise 3.35 demonstrates the technique of using conformal mappings to solve uniform flow problems. Essentially any conformal mapping from a region D onto $\mathbb{H}$ is the complex potential for the velocity field of an ideal fluid flowing in D.

Another example showing the power of conformal mappings is *flow around a cylinder*.

EXAMPLE 3.36. Consider the function $h(z) = z + 1/z$. This function is a conformal mapping from $\mathbb{H}$ minus the upper half of the unit disk onto $\mathbb{H}$. More precisely, h maps $\{z \,|\, |z| \geq 1 \text{ and } \operatorname{Im} z > 0\}$ onto $\mathbb{H}$. The streamlines are shown in Figure 3.11 and are given explicitly by the family of curves

$$y - \frac{y}{x^2 + y^2} = c.$$

3.6. Sources and Sinks

Having seen how to solve for uniform flows in various regions, we now turn our attention to regions that have sources or sinks at various locations on the boundary of

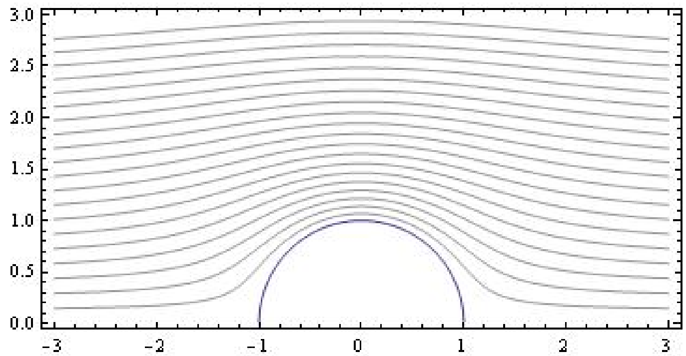

FIGURE 3.11. Streamlines around a Cylinder

the region. Recall the example of the channel from the introduction. Before tackling problems involving channels or other regions, we will first see how to solve the problem in $\mathbb{H}$. Here we will allow any number of sources or sinks of various strength along the real axis.

Let $a \in \mathbb{R}$. We have already seen that $\overline{V(z)} = 1/(\bar{z} - a)$ is an irrotational, incompressible vector field with a source at $z = a$ of strength 2π. However, it is important to note that we are only interested in the part of $V(z)$ that lies in $\mathbb{H}$. Since this is exactly half of the field, it makes sense to define the *effective strength* of the source on the boundary to be half of the strength. In this case, the effective strength of the source is π. As we have already mentioned, only the relative strengths of the sources or sinks matter with regard to the qualitative behavior of the flow. However, if we wish to balance a source on the boundary with a sink in the interior of the region, then the notion of effective strength comes into play.

Before proceeding, it is important to revisit the idea of a source in vector field. The following exploration shows how physical principles lead to our complex function representation of a source or sink.

EXPLORATION 3.37. Suppose that we have an irrotational and incompressible vector field defined on the punctured plane with a source of strength S at $z = 0$. Furthermore, assume that that the flow lines are directed radially away from the origin. We wish to determine a formula for the field. Let $\vec{F}(x, y)$ denote the vector field at all points other than the origin. We assumed the direction of $\vec{F}$ is always directly away from the origin, so the only thing to determine is the magnitude of $\vec{F}$ at each fixed distance from the origin. Since $\vec{F}$ is incompressible, the amount of fluid crossing into any closed region must equal the amount leaving the region. We apply this principle to circles centered at the origin. We interpret the source having strength S to mean that there are S units of fluid entering the region per unit time. Now consider the circle $x^2 + y^2 = R^2$. There must be S units of fluid crossing this circle per unit time. On the other hand, the amount crossing the circle must be equal to the magnitude of the vector field multiplied by the length of the circle. Setting these two quantities equal to each other gives

$$S = |\vec{F}(x, y)| \cdot 2\pi R. \tag{31}$$

Solve this equation for the magnitude of the field and combine the result with the fact that the direction is radial to find an explicit formula for the field.

Now show that the complex representation of the vector field is given by $\overline{V(z)} = S/2\pi\bar{z}$. Hint: $1/\bar{z} = z/|z|^2$.

Immediate consequences of exploration (3.37) are the following results that we have already established:

(1) If the source is located at $z = a$ instead of $z = 0$, then the complex representation of the vector field is given by $\overline{V(z)} = 1/(\bar{z} - \bar{a})$.
(2) If a sink is desired at $z = a$, then the function is given by $\overline{V(z)} = -1/(\bar{z} - \bar{a})$
(3) The strength of a source or sink is changed simply by multiplying $V(z)$ by a real number.

EXERCISE 3.38. Find the complex representation of the irrotational, incompressible vector field in $\mathbb{H}$ if there is a source of effective strength π at $z = 3$ and a sink of effective strength 2π at $z = -2$. Hint: The vector field at any point is the sum of the vector fields determined from each individual source or sink. Use the FlowTool applet to view the solution. ***Try it out!***

Recall that if $\overline{V(z)} = 1/(\bar{z} - \bar{a})$, then the complex potential function on $\mathbb{H}$ can be obtained from $V(z)$ by using Equation (30):

$$\Omega(z) = \int V(z)\, dz = \mathrm{Log}(z - a). \tag{32}$$

Thus, we see that the complex logarithm is involved whenever there are sources and sinks along the boundary. It is important to remember that the complex logarithm extends the real logarithm function. Observe that $\Omega(z)$ is analytic for $\operatorname{Im} z > 0$, but it is not analytic in any domain containing $z = a$.

We recall that

$$\mathrm{Log}(z - a) = \mathrm{Log}|z - a| + i\,\arg(z - a).$$

Thus, the stream lines are given by the family of equations

$$\arctan\left(\frac{y}{x - a}\right) = c.$$

This in turn is easily manipulated into the family of rays emanating from $z = a$ given by $y = (\tan c)(x - a)$. Observe that the values $c = 0$ and $c = \pi$ correspond to the two stream lines that run along the boundary of $\mathbb{H}$ in the positive and negative directions from the source at $z = a$.

The proceeding development allows us to write down the complex potential for an irrotational, incompressible vector field in $\mathbb{H}$ with any combination of sources and sinks along the boundary. We state the complete result in the following theorem.

THEOREM 3.39. Let $\overline{V(z)}$ be the irrotational, incompressible vector field in $\mathbb{H}$ generated by a finte collection of simple sources or sinks of various strengths. Assume there are sources located along the boundary of $\mathbb{H}$ at $z = a_i$, $i = 1, \ldots, n$, $a_i \in \mathbb{R}$, with corresponding effective strengths, $S_i\pi$, $i = 1, \ldots, n$ (note, if $S_j < 0$, then there is a sink at a_j). Then the complex potential of $\overline{V(z)}$ is given by

$$\Omega(z) = \sum_{j=1}^{n} S_j \mathrm{Log}(z - a_j).$$

The following exercise asks you to experiment with the *FlowTool* Applet to develop some intuition regarding flows in $\mathbb{H}$. Feel free to expand beyond the suggestions in the exercise and to experiment with a variety of situations. In each case, the result should make physical sense.

EXERCISE 3.40. Use the *FlowTool* applet to view the flow in $\mathbb{H}$ for the following situations:

(1) A source of strength 2π at $z = -1$ and a sink of strength 2π at $z = 1$.
(2) A source of strength 2π at $z = -1$ and a source of strength 2π at $z = 1$.
(3) A source of strength 4π at $z = -1$ and a sink of strength 2π at $z = 1$.
(4) Sources of strength 2π at $z = -3$ and $z = 0$ and a sink of strength 4π at $z = 2$. ***Try it out!***

You should observe in Exercise 3.40 that in some cases all the fluid emanating from the sources is taken in by the sinks, but in other cases some of the fluid either escapes to infinity or seems to emanate from infinity. If we think of infinity as a point on the boundary of $\mathbb{H}$, then we are led to the notion that there must be a sink or source at infinity.

In order to further examine the behavior at infinity, we revisit a flow in the entire plane with a finite collection of sources or sinks. Specifically, consider the vector field

$$\overline{V(z)} = \sum_{j=1}^{n} \frac{S_j}{2\pi} \frac{1}{\bar{z} - \bar{z}_j}.$$

Let C be a simple closed contour that encloses all the singularities of $V(z)$ and consider $\int_C V(z)\,dz$. By the residue theorem,

$$\mathrm{Im} \int_C V(z)\,dz = \sum_{j=1}^{n} S_j,$$

which gives the total flux across C. Now, $\int_C V(z)\,dz$ can also be viewed as a line integral "around infinity" traversed in the clockwise direction. When viewed in this way, the value of the integral is the opposite sign of the total flux outward across C. For example, if $V(z)$ has a source of strength 3 and a sink of strength 1 at points in the plane, then there must a sink of strength 2 at infinity. Observe that if the net flux in the finite plane is zero, then the sum of the sinks and sources at infinity must be zero as well. The idea that infinity can be both a source and a sink will be explored further as we progress through the chapter. In particular, Section 3.7 investigates flow in an infinite channel, where the concept of infinity as a simultaneous source and sink plays a critical role.

3.7. Flow in a Channel

Here we return to the example of a uniform flow in an infinitely long channel described in the introduction. However, before dealing with any sources or sinks along the boundary, we first investigate the case of uniform flow to the right in the channel. Assume the channel is oriented with one edge along the real axis and the other edge along the line $\operatorname{Im} z = \pi$. Since the edges of the channel are horizontal, the flow is simply the restriction of the uniform flow to the right in $\mathbb{H}$ restricted to the channel. Hence the complex potential is $\Omega(z) = z$.

On the other hand, the function $f(z) = e^z$ is a conformal mapping of the strip onto $\mathbb{H}$ such that boundary edges of the strip are mapped to the real axis. Hence, proceeding as in the case of the quadrant (flow around a corner), and using Theorem 3.32, we see that $\widetilde{\Omega}(z) = \Omega(f(z)) = e^z$ should be the complex potential function for the uniform flow in the strip. However, note that $\operatorname{Im} \widetilde{\Omega}(z) = e^x * \sin y$. Plotting the level curves of this function gives the streamlines shown in Figure 3.12. The streamlines should simply go from left to right parallel to the edges of the strip. What went wrong?

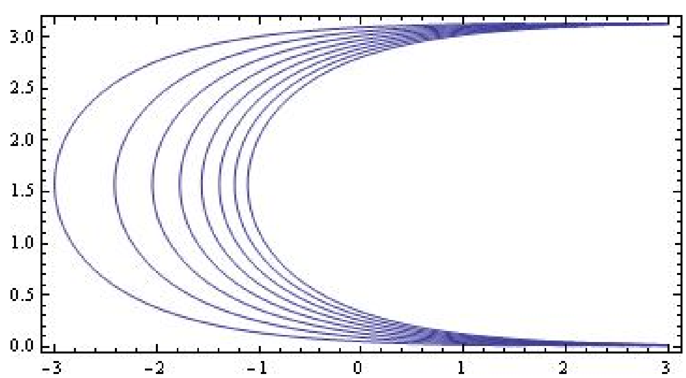

FIGURE 3.12. Incorrect uniform flow in a channel

The incorrect flow occurred from a failure to account for the behavior of infinity under the conformal mapping. Indeed, we may think of the flow in the channel as having a source at the "left end" of the strip and a sink of equal strength at the "right end" of the strip. Now observe that points in the strip with very large negative real parts are mapped near zero. In fact, for all z with $0 \le y \le \pi$, we have

$$\lim_{x \to -\infty} e^{x+iy} = 0.$$

Thus, under the conformal mapping e^z, we expect to see a source at zero in $\mathbb{H}$. On the other hand, as $x \to \infty$, the values of e^z also approach ∞ and the sink is mapped to infinity in $\mathbb{H}$. This means that our complex potential in $\mathbb{H}$ has a source at zero. The complex potential for this flow in $\mathbb{H}$ is $\Omega(z) = \mathrm{Log}(z)$. Thus, the complex potential for the uniform flow in the strip is given by Theorem 3.32 as $\widetilde{\Omega}(z) = \Omega(e^z) = \mathrm{Log}(e^z) = z$ as expected.

With an understanding of how sources and sinks at infinity must be accounted for, it is easy to incorporate sources and sinks along the boundary. The following exercise asks you to find the complex potential for various flows in a channel. You should note in each case that the sum of all sources and sinks must be zero. Use the *FlowTool* applet to see the streamlines for each different flow and to note the behavior at infinity.

EXERCISE 3.41. Consider an infinitely long channel of width π having its lower edge along the real axis. Suppose there is a source of strength 6π at $z = 0$ and a sink of strength 6π at $-3 + \pi i$. Find the complex potential for this flow. Use the FlowTool applet to visualize the streamlines of the flow. Does any of the flow "escape" to infinity? Why or why not? ***Try it out!***

EXERCISE 3.42. Consider a similar problem to Exercise 3.41, but now suppose that the sink at $-3 + \pi i$ has strength 2π. Carefully describe the behavior at infinity. Find the complex potential and use the FlowTool applet to plot the streamlines. Does the result make sense from a physical perspective? ***Try it out!***

3.8. Flows in Other Regions

The methods used to solve flow problems in a strip apply equally well to any other region with sources or sinks along the boundary. All that is required is a conformal mapping from the region in question to the upper half-plane $\mathbb{H}$, such that the boundary of the region is mapped to the real axis. It is then a matter of determining where the sources and sinks are mapped and accounting for any behavior at infinity. It is then straightforward to find the complex potential for the transformed problem in $\mathbb{H}$ and then compose the result with the conformal mapping to obtain the solution.

EXAMPLE 3.43. As an example of these techniques, consider the region $R = \mathbb{H} - \{z \,|\, |z| \le 1\}$. Imagine a uniform flow in R going left to right, along with a source of strength 2π at $z = -1$ and a sink of strength 2π at $z = 1$. We find the complex potential for this flow in R as follows:

(1) The mapping $f(z) = z + 1/z$ maps R conformally to $\mathbb{H}$.
(2) Observe that $f(-1) = -2$ and $f(1) = 2$.
(3) The complex potential in $\mathbb{H}$ is $\Omega(z) = \mathrm{Log}(z + 2) - \mathrm{Log}(z - 2)$.

(4) Composing Ω with f gives the result: $\widetilde{\Omega}(z) = \mathrm{Log}(z{+}1/z{+}2) - \mathrm{Log}(z{+}1/z{-}2)$.

(5) The flow lines are given by the level curves of $\mathrm{Im}\,\widetilde{\Omega}$.

The flow lines are shown in Figure 3.13.

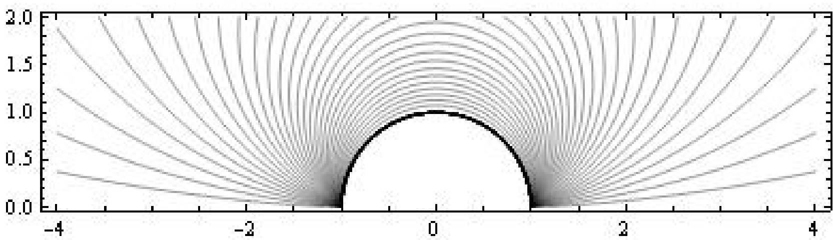

FIGURE 3.13. Flow around cylinder with sources and sinks

Apply the methods outlined in the example above in the following exercise.

EXERCISE 3.44. Let $R = \{z \,|\, 0 \le \arg(z) \le \pi/3\}$. Assume there is a source of strength 4π at $z = 0$ and a sink of strength 2π at $z = 2$. Find complex potential for the flow in R. The stream lines are shown in Figure 3.14

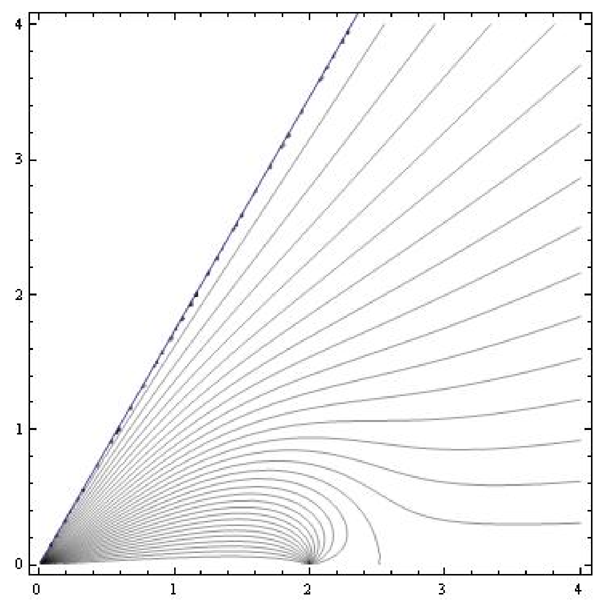

FIGURE 3.14. Stream lines for Exercise 3.44

It should be noted that even with a sophisticated graphing program it may not be easy to obtain a sketch of the flow lines. Many of the plots in this chapter have been generated using *Mathematica*. The *Mathematica* code used to generate Figure 3.14 is shown below.

```
>F[z_]=4*Log[z^3]-2*Log[z^3-8]
>a=ContourPlot[If[y<Sqrt[3]*x, Im[F[x+I*y]]], {x, 0, 4}, {y, 0, 4},
   ContourShading->False, Contours->30]
>b=Plot[Sqrt[3]*x, {x, 0, 4}]
>Show[a, b]
```

In the *Mathematica* code, observe the use of the "If" statement to only plot stream lines in the region of interest. The second plot, labeled "b" is technically not needed, but it makes the boundary of the region much clearer in the final plot.

A few more comments regarding the use of *Mathematica* are in order. In order to make a contour plot of the real or imaginary part of a complex function, it is necessary to replace z by $x + i\,y$ in the plot command. In addition, if the various branch cuts for complex logarithm functions may result in a plot with some breaks in the flow lines that does not look quite right. Sometimes it is possible to fix this problem by combining all the logarithm terms. For example in the code fragment above, the function could have been defined as

```
F[z_]=Log[z^12/(z^3-8)^2]
```

The *FlowTool* applet provides the first quadrant as one of the domains available for study. In the absence of any sources or sinks on the boundary, the uniform flow in this region is simply the "flow around a corner" that we have already mentioned. By including sources or sinks some some very interesting flows can be seen.

EXPLORATION 3.45. Develop a general procedure for finding the complex potential for a flow in the first quadrant with various sources or sinks on the boundary. Use your method to find the complex potentials for the scenarios given below. In each case, use the *FlowTool* applet to investigate the flow.

(1) Sources of equal strength at $z = 1$ and $z = i$.
(2) A source and a sink of equal strengths at $z = 1$ and $z = i$.
(3) Sources of equal strength at $z = 1$ and $z = i$, and a sink of double strength at the origin.

3.9. Flows inside the Disk

The methods developed thus far can be applied equally well to fluid flow inside a disk. The following set of exercises explores various scenarios. Obtaining the graphical output for these flows can be challenging. We present the *Mathematica* code to generate one of the plots.

EXERCISE 3.46. Let $\mathbb{D} = \{z \in \mathbb{C} \,|\, |z| < 1\}$. Show that

$$f(z) = i\,\frac{1-z}{1+z}$$

is a conformal mapping from the unit disk $\mathbb{D}$ to the upper half plane $\mathbb{H}$. Determine the images of the points ± 1 and $\pm i$. Use the conformal mapping to find a uniform flow inside $\mathbb{D}$. The flow lines are shown in Figure 3.15. Explain why there appears to be both a source and a sink at $z = -1$. ***Try it out!***

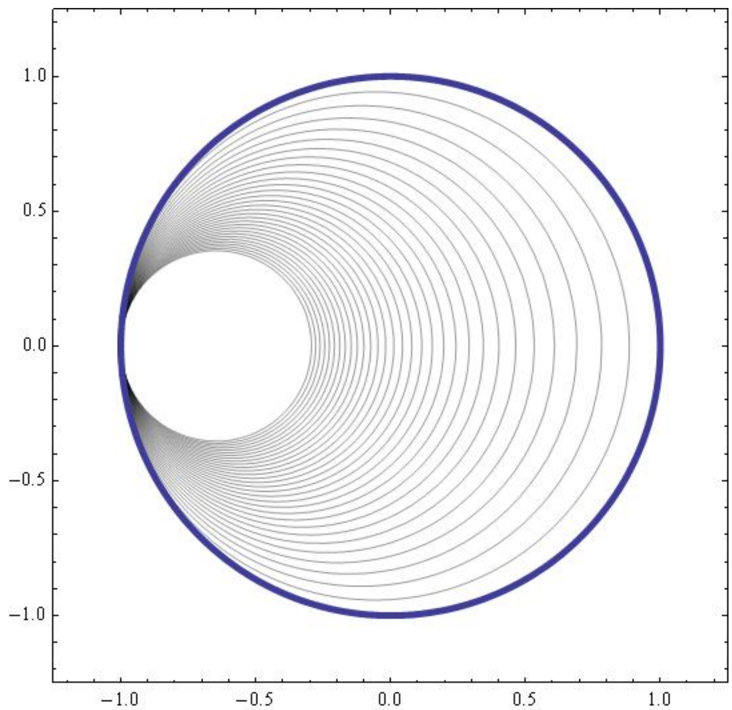

FIGURE 3.15. Uniform flow in a disk. Observe the behavior at $z = -1$.

The *Mathematica* code for Figure 3.15 is given below.

```
>g[z_] = I*(1 - z)/(1 + z)
>M[x_, y_] = If[x^2 + y^2 <= 1, Im[g[x + I y]]]
>a = ContourPlot[M[x, y], {x, -1.2, 1.2}, {y, -1.2, 1.2},
  ContourShading -> False, Contours -> 30]
>b = ParametricPlot[{Cos[t], Sin[t]}, {t, 0, 2*Pi},
  PlotStyle -> Thickness[.01]]
>Show[a, b]
```

EXPLORATION 3.47. This exploration considers various flows inside a disk.

(1) Find the velocity field of the ideal fluid flow inside $\mathbb{D}$ if there is a source of strength 2π located at $z = 1$. Note that the total strength of the sources in a problem must be balanced by a sink or sinks of equivalent strength. Where is the sink in this problem? Can you control the location of the sink?

(2) Find the complex potential for the flow in $\mathbb{D}$ if there is a sink of strength 2π at $z = i$.
(3) Consider the case where there are equal strength sources on the boundary of $\mathbb{D}$ located at $z = 1$ and $z = -1$. Consider the location of the resulting sink. Why does this occur?

3.10. Interval Sources and Sinks

In this section we extend some of the methods already developed. Instead of looking at point sources or sinks, we consider the case of an entire interval of sources or sinks. This will allow us to model different phenomenon such as flow through a levy or the electric field generated by a line of charges. We present the material as a series of exercises designed to lead to the key results. The reader is expected to complete these exercises rather than simply using the results.

We begin by examining the flow lines generated by a uniformly distributed source of total strength 2π located along the interval $a \leq x \leq b$.

We expect to see a vector field as in Figure 3.16.

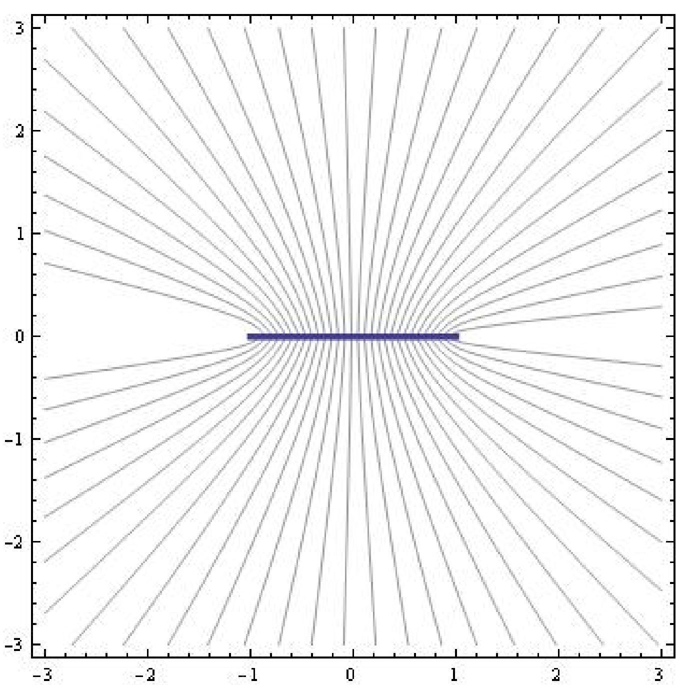

FIGURE 3.16. Vector field with a uniform interval source

The next set of exercises shows that the complex potential for an interval source can be derived as a limiting process of a collection of point sources as the number of points approaches infinity.

EXERCISE 3.48. Let $a = x_0, x_1, \ldots, x_n = b$ be $n+1$ equally spaced points in the interval $[a,\, b]$. Assume there is a source of strength $2\pi/(n+1)$ at each x_j. Find the complex potential $\Omega(z)$ for the flow in $\mathbb{H}$ for this collection of point sources. ***Try it out!***

Now the idea at this point is to let $n \to \infty$ and to recognize the resulting limit as a definite integral.

EXERCISE 3.49. Let $\Delta x_j = x_{j+1} - x_j$, $j = 0, 1, \ldots, n$. Express the result of the previous exercise as a Riemann sum on the interval $[a,\, b]$. Show the limit of the Riemann sum leads to the definite integral

$$\frac{1}{b-a}\int_a^b \mathrm{Log}(z-x)\,dx. \tag{33}$$

Try it out!

EXERCISE 3.50. Obtain the definition of the complex potential for a uniform interval source by using integration by parts to show the that the integral 33 can be evaluated to the expression:

$$\Omega_a^b(z) = \frac{b-z}{b-a}\mathrm{Log}(z-b) + \frac{z-a}{b-a}\mathrm{Log}(z-a) - 1 \tag{34}$$

Try it out!

EXERCISE 3.51. Use Equation (34) to find the complex potential in $\mathbb{H}$ for a flow that has a uniform source of strength 2π along the interval $[0,\, 3]$ and a uniform sink of strength 2π along the interval $[-2,\, -1]$. The graph appears in Figure 3.17. ***Try it out!***

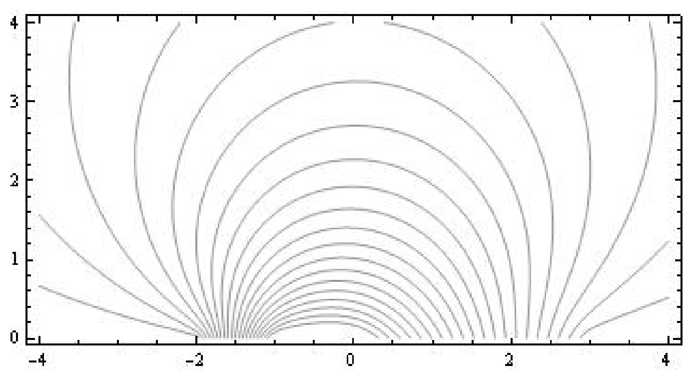

FIGURE 3.17. Flow lines with interval source/sink

Next we seek to extend the notion of interval sources or sinks to the boundaries of other regions. Some care must be taken to be sure the interval sources or sinks are still uniform as we now show. Consider the case of the first quadrant R in $\mathbb{C}$

with interval sources of equal strength located along the intervals $[1,\,3]$ and $[i,\,3i]$. We proceed as in previous sections. The conformal mapping $f(z) = z^2$ maps R to $\mathbb{H}$. Next we determine the behavior of the key intervals under the conformal mapping. The interval $[1,\,3]$ is mapped to $[1,\,9]$ and $[i,\,3i]$ is mapped to $[-9,\,-1]$. Now, we know how to find the complex potential in $\mathbb{H}$ from Equation (34). Indeed we see that $\Omega(z) = \Omega_1^9(z) + \Omega_{-9}^{-1}(z)$. We then find the complex potential in R by composition: $\widetilde{\Omega}(z) = \Omega(z^2)$. Unfortunately this result is not quite correct. The problem is that that sources should be uniformly distributed across the intervals in R. When we apply the conformal mapping the resulting intervals in $\mathbb{H}$ are no longer uniformly distributed. The key to fixing this problem is described next.

To understand what is happening, focus on the interval $[1,\,3]$ under the mapping $f(z) = z^2$. Clearly the interval is mapped to $[1,\,9]$. Notice that the first half $[1,\,2]$ is mapped to $[1,\,4]$ and the second half $[2,\,3]$ is mapped to $[4,\,9]$. Hence the density of points is less in the second half of the interval. Thus, when solving for the complex potential in $\mathbb{H}$ we cannot treat the intervals as though the source is uniformly distributed. Recall that Equation (34) was derived under the assumption that the source was uniformly distributed along the interval. We must instead use Equation (33) and take into account the non-uniform distribution. Thus instead of computing the integral $\int_1^9 \operatorname{Log}(z-x)\,dx$ we compute $\int_1^3 \operatorname{Log}(z-x^2)\,dx$.

Exercise 3.52. Find a complex potential for the flow in the first quadrant generated by uniform interval sources along the intervals $[1,\,3]$ and $[i,\,3i]$. Plot the streamlines.

Exercise 3.53. Find the complex potential for an ideal flow in the region

$$R = \{re^{i\theta} \,|\, r \geq 0, \quad 0 \leq \theta \leq \frac{\pi}{3}\}$$

with a uniform source of strength 2π located along the interval $[2,\,4]$. ***Try it out!***

Exercise 3.54. Find the complex potential for an ideal flow in the infinite channel

$$R = \{z \,|\, 0 \leq \operatorname{Im} z \leq 2\}$$

with a uniform source of strength 2π located on the boundary along the interval $[1 + 2i,\,4 + 2i]$. ***Try it out!***

A natural extension of the previous material on interval sources is to consider intervals with a non-uniform density. This small project was inspired by Potter [**4**].

Small Project 3.55. Consider a function $\lambda : [a,\,b] \subset \mathbb{R} \to \mathbb{R}$ with the property that

$$\int_a^b \lambda(x)\,dx = S,$$

where S is the total strength of the generalized source on $[a,\,b]$. We think of $\lambda(x)$ as giving the source density at $x \in [a,\,b]$. The goal is to find the complex potential for this interval source with variable density. Observe that if $\lambda(x) = S/(b-a)$, for all x in $[a,\, b]$, then we obtain the case of a uniformly distributed source along the interval.

Rather than attempting to directly find the complex potential, it is better to first find the underlying vector field $\overline{V(z)}$. Begin by subdividing the interval into n equal subintervals, where the n^{th} subinterval is $[x_i,\, x_{i+1}]$. Now, consider the vector field $\overline{V_i(z)}$ having a source at x_i with strength

$$S_i = \frac{b-a}{n}\lambda(x_i).$$

Show by summing the individual vector fields and taking a limit as $n \to \infty$ that the vector field is given by

$$\overline{V(z)} = \int_a^b \frac{\lambda(x)}{\bar{z} - x}\, dx. \tag{35}$$

From Equation (35) it follows that the complex potential is

$$\Omega(z) = \int \left(\int_a^b \frac{\lambda(x)}{z - x}\, dx \right) dz \tag{36}$$

Consider the specific example of $\lambda(x) = x$ on the interval $[-1,\, 2]$. Observe that the source density takes on both positive and negative values on the interval, with the positive density increasing towards the right end of the interval. The strength of this variable density source is

$$\int_{-1}^{2} x\, dx = \frac{3}{2}.$$

Compute $V(z)$ by using the substitution $w = z - x$ in the integral. Use the result to find the complex potential and plot the flow lines. The result is shown in Figure 3.18.

Note that if λ is a polynomial, then a formula for $V(z)$ can be found explicitly using the same substitution as above. The reader is encouraged to find and plot the complex potential for the case where $\lambda(x) = x^2$ on $[0,\, 1]$.

3.11. Steady State Temperature Problems

A common problem in applied mathematics is to find the steady state temperature in a region of the plane given a fixed temperature distribution along the boundary of the region. In the case of the upper half plane $\mathbb{H}$, the problem takes the following form: Given a piecewise continuous, bounded function f defined on the real axis (i.e., the temperature), find a function T in $\mathbb{H}$ such that $\Delta T = 0$ and T "agrees" with f along the boundary. Note that solution involves finding a harmonic function in $\mathbb{H}$. We also point out that the we are seeking solutions that are physically meaningful. For example, if the boundary values are identically zero, then both $T_1(x,\, y) = 0$ and $T_2(x,\, y) = e^x \sin y$ are solutions in $\mathbb{H}$. However, T_2 is unbounded which does not correspond to a real temperature distribution.

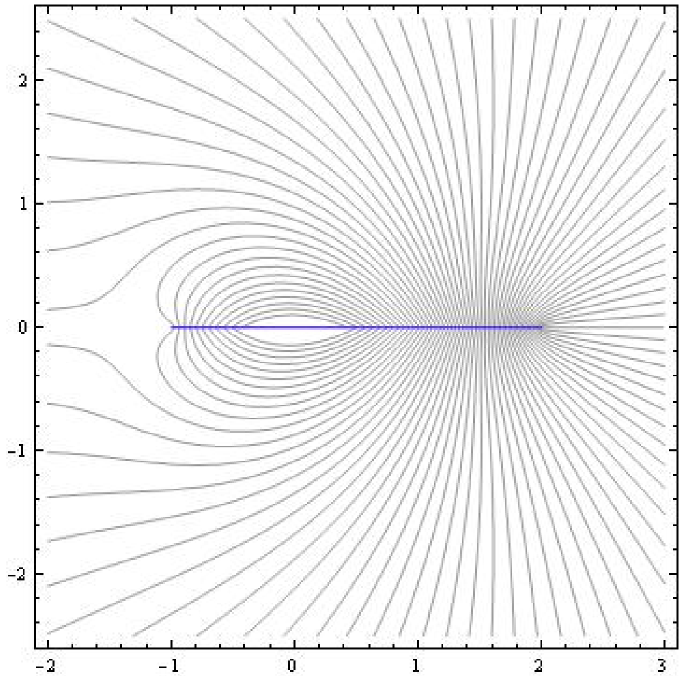

FIGURE 3.18. Flow lines for variable density interval source

Every time we construct a complex potential function in $\mathbb{H}$, both the real and imaginary parts are harmonic. To see if our methods are useful in this setting, we discuss how to manipulate the boundary values in an example.

We begin with the simple case of the complex potential for an ideal flow in $\mathbb{H}$ with a single point source of strength 2π at $z = 0$. We know that the complex potential is given by $\Omega(z) = \text{Log } z$. Writing this in terms of its real and imaginary parts gives

$$\Omega(z) = \text{Log } |z| + i \text{ Arg}(z). \tag{37}$$

Now observe the values of $\text{Arg}(z)$ along the real axis. When $x > 0$, we have $\text{Arg}(x) = 0$, and when $x < 0$ we have $\text{Arg}(z) = \pi$. The key observation is that the argument function is piecewise constant along the real axis.

Consider the steady state temperature problem where the distribution along the boundary is given by

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 100 & \text{if } x \geq 0. \end{cases} \tag{38}$$

Starting from Equation (37), we see that multiplying $\text{Im}\,\Omega(z)$ by $100/\pi$ gives a harmonic function in $\mathbb{H}$ satisfying the boundary condition

$$\frac{100}{\pi}\operatorname{Arg}(x) = \begin{cases} 100 & \text{if } x < 0 \\ 0 & \text{if } x \geq 0. \end{cases} \tag{39}$$

This is almost correct. To get the boundary values switched around to the correct intervals we need some properties of the argument function. The following exercise leads to the correct result.

EXERCISE 3.56. If $\operatorname{Im} z > 0$, then show that reflecting $\operatorname{Arg}(z)$ across the y-axis gives the function $\operatorname{Arg}(-\bar{z})$. Next, show that

$$\operatorname{Arg}(-\bar{z}) = -\operatorname{Arg}(-z).$$

Try it out!

Using the results of Exercise 3.56, we see that

$$g(x, y) = -\frac{100}{\pi}\operatorname{Arg}(-z)$$

is harmonic and satisfies the correct boundary conditions. The graph of $g(x, y)$ is shown in Figure 3.19. Note that $g(x, y)$ is the imaginary part of the complex potential

$$\Omega(z) = -\frac{100}{\pi}\operatorname{Log}(-z).$$

The streamlines correspond to isotherms–curves of constant temperature.

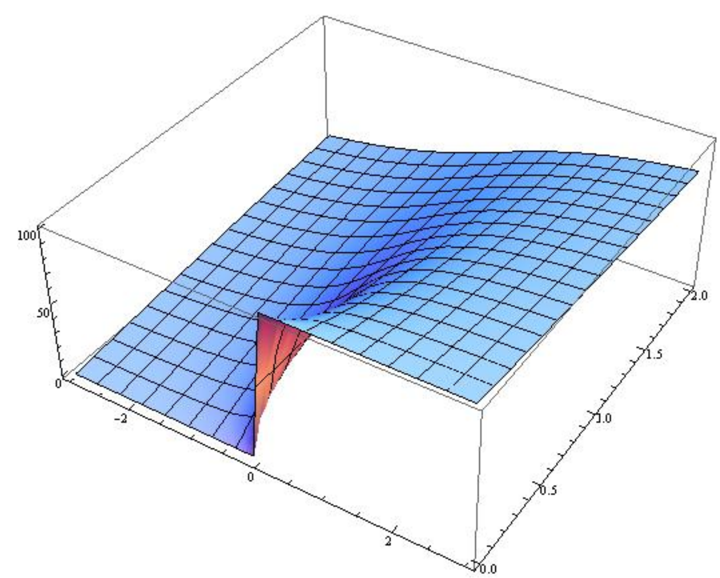

FIGURE 3.19. Steady state temperature distribution with piece-wise constant boundary values

The next few exercises explore slight variations on this problem. We then finish the chapter by extending to the case where segments of the boundary are not insulated (i.e., there are segments that have non-constant temperatures) and see how our methods involving intervals sources can be applied.

EXERCISE 3.57. Solve the steady state temperature problem in $\mathbb{H}$ where the boundary temperatures are given below. (Hint: This is very similar to the above example–simply start with a slightly different complex potential function.)

$$f(x) = \begin{cases} 100 & \text{if } x < 2 \\ 0 & \text{if } x \geq 2. \end{cases} \tag{40}$$

Try it out!

EXERCISE 3.58. Solve the steady state temperature problem in $\mathbb{H}$ where the boundary temperatures are given below. (Hint: Add a constant to the function, but pay attention to the effect it has on the other boundary value.)

$$f(x) = \begin{cases} 100 & \text{if } x < 2 \\ 50 & \text{if } x \geq 2. \end{cases} \tag{41}$$

Try it out!

The types of boundary conditions we have been considering can easily be extended to more segments. The following exercise leads to the algorithm for solving the general problem of this type.

EXERCISE 3.59. Solve the steady state temperature problem in $\mathbb{H}$ where the boundary temperatures are given below.

$$f(x) = \begin{cases} 100 & \text{if } x < -3 \\ 50 & \text{if } -3 < x < 2 \\ 25 & \text{if } x \geq 2. \end{cases} \tag{42}$$

Make a contour plot of your solution and observe that the contours represent curves of constant temperature. ***Try it out!***

The general problem of finding the temperature distribution in the half-plane given piece-wise constant boundary conditions is a standard application in complex analysis texts; see [**3**]. The problem usually takes the following form; see [**3**].

EXPLORATION 3.60. Solve:

$$T_{xx} + T_{yy} = 0$$

Subject to:

$$T(x,0)=\begin{cases} k_0 & \text{if } -\infty < x < x_1 \\ k_1 & \text{if } x_1 < x < x_2 \\ \vdots & \vdots \\ k_n & \text{if } x_n < x < \infty \end{cases}$$

Using the techniques developed in the exercises, derive the general solution to this problem.

Next we consider a slightly more general problem where segments of the boundary are not kept at constant temperature. For example, suppose we have the following temperature distribution along the boundary:

$$f(x)=\begin{cases} 100 & \text{if } x<0 \\ 0 & \text{if } x>1. \end{cases} \tag{43}$$

Observe that no temperature is specified on the interval (0, 1). If this segment of the boundary is not insulated then we expect the temperature to change linearly from 100 degrees to 0 degrees (this can be seen by solving the one-dimensional heat flow problem on the interval [0, 1] with the endpoints held at 100 and 0 degrees respectively).

We now investigate how our earlier techniques can be applied to this type of problem. Let us attempt to find the steady state temperature distribution in $\mathbb{H}$ for the boundary condition given in Equation (43). We begin by writing down the complex potential for an ideal flow with an interval source uniformly distributed along the interval [0, 1]. Recall from Equation (34) that the complex potential is given by

$$\Omega_0^1(z)=(1-z)\mathrm{Log}(z-1)+z\mathrm{Log}(z)-1. \tag{44}$$

We know that $\operatorname{Im}\Omega_0^1(z)$ must be constant along the intervals $x<0$ and $x>1$. We determine these values by choosing a test value from each interval. Thus,

$$\operatorname{Im}\Omega_0^1(-1)=\operatorname{Im}(2\mathrm{Log}(2)-\mathrm{Log}(-1)-1)=-\pi,$$

and

$$\operatorname{Im}\Omega_0^1(2)=\operatorname{Im}(-\mathrm{Log}(1)+2\mathrm{Log}(2)-1)=0.$$

Hence multiplying our complex potential by $-100/\pi$ gives the correct temperatures on the two intervals that are held constant. But, what about the interval (0, 1)? In figure 3.20, we show the graph of $\operatorname{Im}\Omega_0^1(x)$, for $-3<x<3$ and observe the desired linear behavior on the interval (0, 1).

The graph of the solution on $\mathbb{H}$ is shown in Figure 3.21. Notice how the surface representing the temperature matches up with the given boundary values.

The following small project asks you to generalize the previous example to a general problem of the same type.

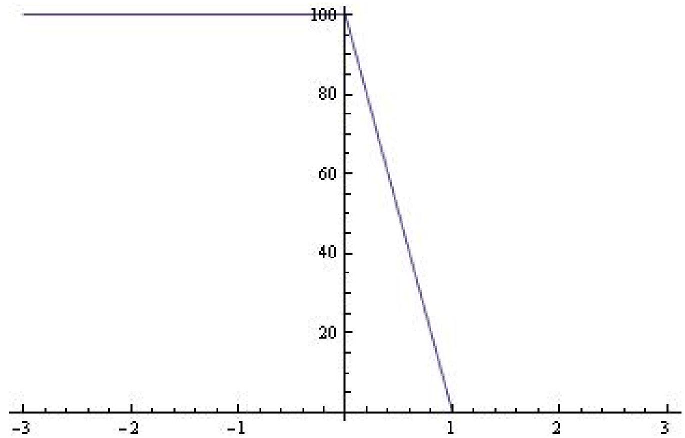

FIGURE 3.20. Temperature along boundary

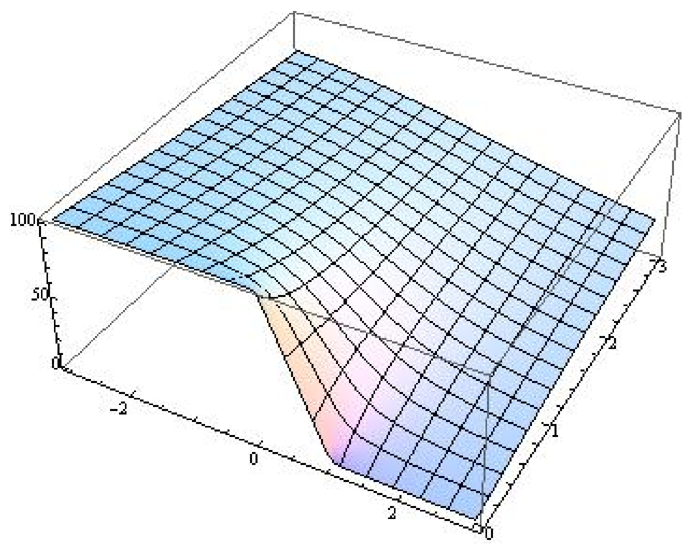

FIGURE 3.21. Solution to heat equation

SMALL PROJECT 3.61. Find a general formula for the steady-state temperature distribution $T(x, y)$ in the half-plane with the following boundary data:

$$T(x,0) = \begin{cases} k_1 & \text{if } -\infty < x < x_1 \\ k_2 & \text{if } x_2 < x < x_3 \\ \vdots & \vdots \\ k_n & \text{if } x_n < x < \infty \end{cases}$$

We assume $x_1 < x_2 < \ldots < x_n$ and that $T(x, 0)$ is linear on the intervals in between the x_i locations.

3.12. Flows with Source and Sinks not on the Boundary

In this section we extend the ideas of this chapter to a wider array of the applications. Consider the standard example of the upper half plane $\mathbb{H}$, but with a source located at $z = i$, as opposed to being on the boundary. In this case, we expect the flow to look like an ordinary source near $z = i$, but since it is constrained to stay in $\mathbb{H}$ the real axis must deflect the flow so that it runs parallel to the boundary. Figure 3.22 shows the flow lines we should expect.

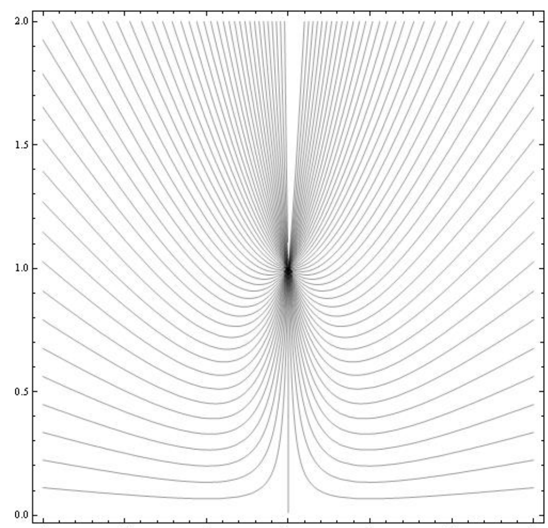

FIGURE 3.22. Flow in $\mathbb{H}$ with a single source at $z = i$.

The key idea in obtaining the complex potential that gives Figure 3.22 is to balance the source at $z = i$ with a source of equal strength at $z = -i$. By the Principle of Superposition, the vertical components of the underlying vector fields will sum to zero along the real axis. The resulting complex potential is $\Omega(z) = \text{Log}(z - i) + \text{Log}(z + i)$ or equivalently as $\Omega(z) = \text{Log}(z^2 + 1)$. Note that Ω is not analytic in on any punctured neighborhood of i. Thus, Ω is not formally a complex potential in all of $\mathbb{H}$. However, by choosing different branches of the complex logarithm, it always possible to construct an analytic complex potential in a simply connected domain not containing i.

This balancing sources or sinks across the boundaries, combined with our earlier work can be used to deal with sources or sinks in the interior of any region. Needham [**2**] refers to this approach as the *Method of Images*. The following exercise finds the complex potential for a flow in the first quadrant with a source located at the interior point $z = 1 + 2i$.

EXERCISE 3.62. Find the flow of an ideal fluid in the first quadrant with a single source located at $z = 1 + 2i$. Hint: Use the conformal mapping $f(z) = z^2$ to map the first quadrant to $\mathbb{H}$. Determine the location of source by computing $f(1 + 2i)$. Find the complex potential in $\mathbb{H}$ by balancing the source with another source symmetrically located across the real axis. Compose the result with $f(z)$ to obtain the complex potential of the flow in the first quadrant. The result gives the flow shown in Figure 3.23. ***Try it out!***

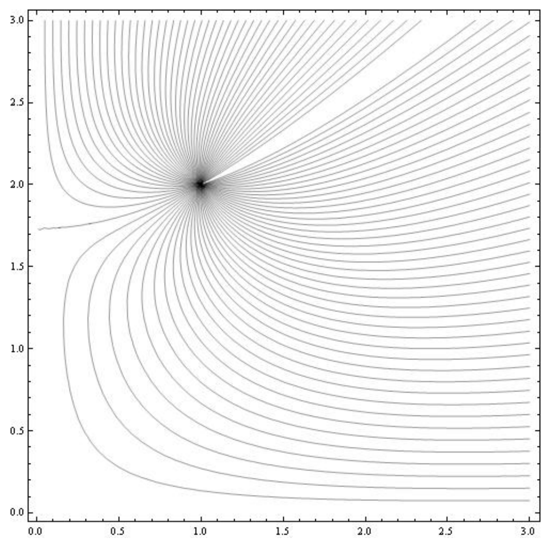

FIGURE 3.23. Flow in the first quadrant with a single source at $z = 1 + 2i$

When dealing with sources and sinks both on the boundary and in the interior the notion of *effective strength* comes into play. Indeed, a source on the boundary must have twice the strength as one in the interior in order to have the same effective strength.

EXERCISE 3.63. Find the complex potential for a uniform flow in $\mathbb{H}$ with a source at $z = 0$ and a sink of equal magnitude at $z = i$. Answer: $\Omega(z) = \operatorname{Log}(z^2/(z^2 + 1))$ The streamlines are shown in Figure 3.24 ***Try it out!***

The techniques developed thus far allow us to combine sources and sinks of various relative strengths both on the boundary and in the interior of some region. The following exercise shows that the complex potential for a very complicated flow can be systematically built up from simpler pieces.

EXERCISE 3.64. Let R be the infinite strip defined by $0 \leq \operatorname{Im} z \leq \pi$. Suppose there is a source of strength 2π at $z = 1$, a sink of strength 2π at $z = 4$ and a source of strength 2π in the middle of the strip at $z = \pi i/2$. Construct the complex potential of the velocity field. The streamlines are shown in Figure 3.25. ***Try it out!***

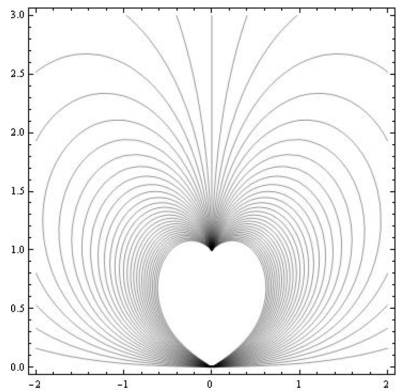

FIGURE 3.24. Balancing a source on the boundary with a sink in the interior

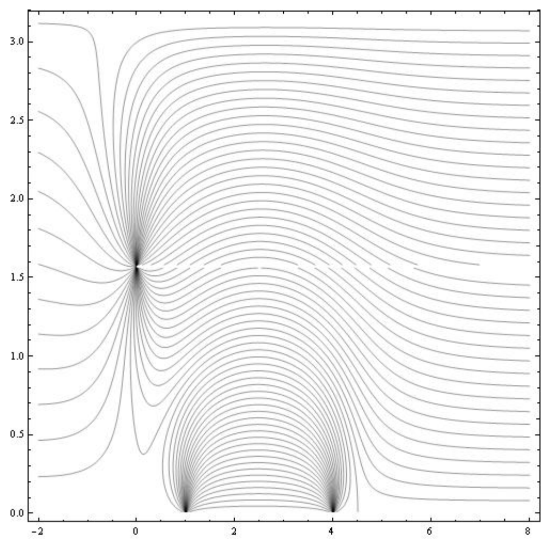

FIGURE 3.25. Streamlines for the flow in Exercise 3.64

3.13. Vector Fields with Other Types of Singularities

Another application arises when we interpret ideal flows as electric field lines generated by various positive and negative charges. In this context we imagine the plane as a copper plate with charges at various locations. A *dipole* is obtained when a positive and negative charge of equal strength are separated by a small distance. Consider the case of a source of strength 1 located at $z = \epsilon$ and a sink of strength -1 located at $z = -\epsilon$. The Polya vector field for the corresponding electric field is

$$\overline{f(z)} = \frac{1}{\bar{z} - \epsilon} - \frac{1}{\bar{z} + \epsilon}.$$

Observe that as ϵ approaches zero, the field vanishes. In order to prevent the field from vanishing, we need to increase the strengths of the source and sink inversely to the distance between them. keep the field strength constant while the distance between the charges approaches zero.

Exercise 3.65. Consider the Polya vector field

$$\overline{f(z)} = \frac{1}{2\epsilon}\left(\frac{1}{\bar{z}-\epsilon} - \frac{1}{\bar{z}+\epsilon}\right).$$

Compute the limiting field as ϵ goes to zero and then compute the complex potential for the field. Hint: Recall that $\Omega'(z) = f(z)$. ***Try it out!***

As shown in Exercise 3.65, the vector field generated by the dipole is $\overline{f(z)} = 1/\bar{z}^2$. Observe the complex potential is given by $\Omega(z) = -1/z$. The electric field lines are given by the level curves of $\operatorname{Im}\Omega(z)$ and are shown in Figure 3.26.

Exercise 3.66. Show that the electric field lines for the dipole are circles with their centers along the imaginary axis.

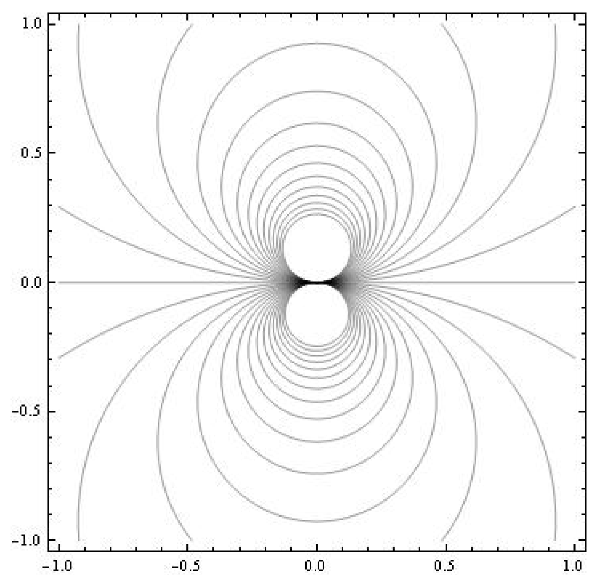

Figure 3.26. Electric field lines for a dipole.

Observe the orientation of the two small loops in Figure 3.26. The particular orientation of the dipole depends on the direction from which the source and the sink approach each other. In this case Polya field was $1/\bar{z}^2$ and the *dipole moment* was 1. Other dipole moments are obtained by considering the field $d/\bar{z}^2$, where d is any complex number. For example if $d = 1 + i$, then we obtain the dipole field shown in Figure 3.27.

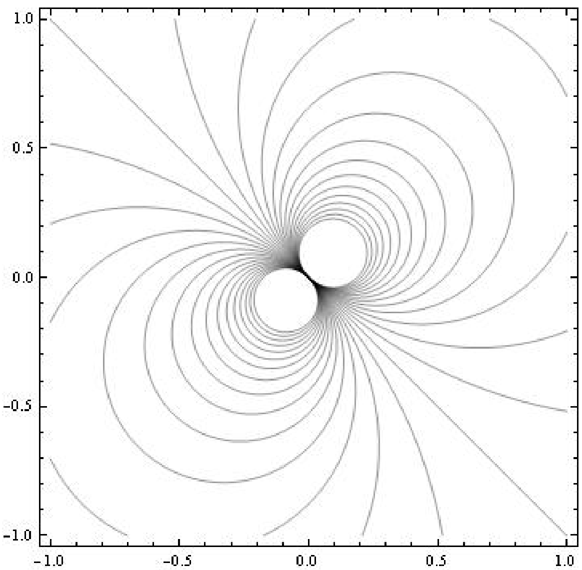

FIGURE 3.27. Electric field lines for a dipole with dipole moment equal to $1+i$.

EXPLORATION 3.67. Investigate the general behavior of the dipole whose field is given by $\overline{f(z)} = (a+bi)/\bar{z}^2$.

It is also possible to consider the problem of multiple charges approaching each other to obtain *multipoles*. As an example, consider the complex potential function $\Omega(z) = 1/z^2$. The electric field lines are shown in Figure 3.28.

EXERCISE 3.68. What charges are converging to give the electric field lines shown in Figure 3.28?

An interesting problem involves looking at sources or sinks and multipoles at the same location. For example, consider the Polya field $\overline{V(z)} = 1/\bar{z}^2 + 1/\bar{z}$. The complex potential function is $\Omega(z) = -1/z + \text{Log } z$. As an exercise, you should plot the electric field lines both near zero and on a larger scale. Note how the dipole dominates the behavior near zero and the source dominates the behavior far from zero.

We close this chapter with a brief discussion that ties together a loose end: namely the nature of sources and sinks at infinity. Since we now understand the idea of a multipole, we are in a position to correctly state the nature of a flow at infinity.

Consider the most basic example of the uniform flow to the right in the entire plane $\mathbb{C}$. Since fluid appears from the left and disappears to the right it seems reasonable to say that there is both a source and a sink at infinity. Of course we now recognize this phenomenon as a dipole.

To study the behavior of a complex function at infinity, it is standard to replace z with $1/z$ and study the behavior of the resulting function at $z = 0$. For the case of the uniform flow, the complex potential is $\Omega(z) = z$. The behavior at infinity is

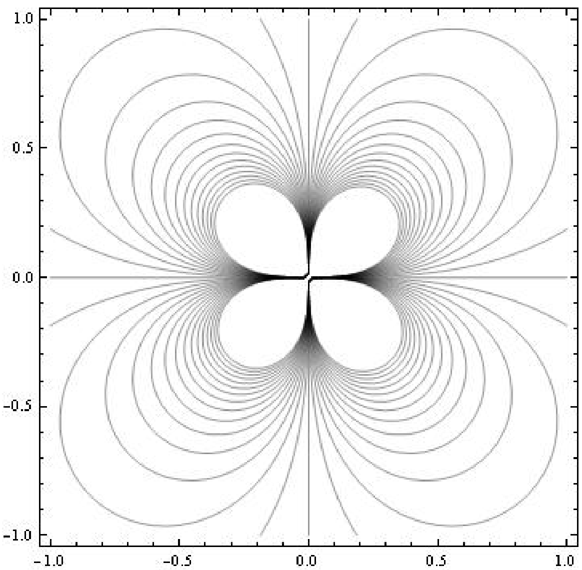

FIGURE 3.28. Electric field lines for a multipole.

determined by examining $\Omega(1/z) = 1/z$ at $z = 0$. Thus we have a potential function of $1/z$. Differentiating and taking the conjugate to obtain the underlying vector field, we get $1/\bar{z}^2$ which we recognize as a dipole.

More generally, for any vector field where the net sum of the sources and sinks is, say N, the strength of the source or sink at infinity is $-N$. If $N = 0$, then there is a multipole at infinity.

Bibliography

[1] James Stewart, *Multivariable Calculus,* 6^{th} *ed.,* Thompson Brooks/Cole, 2008.
[2] Tristan Needham, *Visual Complex Analysis,* Oxford University Press, Oxford UK, 1997.
[3] Dennis G. Zill, Shanahan, P. D., *A First Course in Complex Analysis with Applications,* 2^{nd} *ed.,* Jones and Bartlett, Boston-Toronto-London-Singapore, 2009.
[4] Harrison Potter, *On Conformal Mappings and Vector Fields*, Senior Thesis, Marietta College, Marietta, Ohio, 2008.
[5] Ruel V. Churchill, Brown, J. W., *Complex Variables and Applications,* 4^{th} *ed.,* McGraw-Hill Book Company, New York, 1984.