Complex Variable Problems For Undergraduates

1. Introduction

Remark 1.1. This note is a brief draft intended to show problems that could be analyzed by undergraduates. However, the presentation in this note is intended for a more technical audience and may not be accessible for an undergraduate reader. This is done simply for brevity and would be expanded with more introductory materials and references when presented to an undergraduate. At the same, some motivation to each problem is provided here as well.

Here are the basic complex variables problems that I have considered to which an undergraduate could contribute. I'm assuming a one-semester course in complex variables that would include the statements of the Maximum Modulus Theorem (Possibly Schwarz's Lemma) and the Riemann Mapping Theorem.

- Schwarz-Pick Lemmas for Hyperbolic Derivatives and Schur Coefficients
- Maximal Bieberbach-Eilenberg Functions Which Are Non Self Maps
- (Finite) Rearrangements of Analytic Self Maps of the Unit Disk

Throughout this note, every function f will be analytic within the open unit disk $\mathbb{D} = \{z : |z| < 1\}$. In all but one section, these functions are self maps of the unit disk $|f(z)| < 1$. The Taylor series of f about 0 will be $\sum_{k=0}^{\infty} a_k z^k$.

2. Schwarz-Pick Lemmas for Hyperbolic Derivatives

Along with the standard Euclidean distance, the unit disk can be equipped with an alternative hyperbolic distance which is described below.

Definition 2.1. For a given complex number $a \in \mathbb{D}$, define the automorphism $\phi_a(z) = \frac{a-z}{1-\bar{a}z}$.

Definition 2.2. The **hyperbolic distance** ρ between two points $z, w \in \mathbb{D}$ is

$$\rho(w, z) = \rho(0, \phi_w(z)) = \frac{1}{2} \ln \left(\frac{1 + |\phi_w(z)|}{1 - |\phi_w(z)|} \right)$$

The following sequence of lemmas (without proof) indicate how Schwarz's lemma has evolved in relation to the hyperbolic metric.

Lemma 2.3 (Schwarz's Lemma)**.** *If f is an analytic self-map of the unit disk $\mathbb{D}$,* **and** $f(0) = 0$, *then*

- $|f(z)| \leq |z|$, *and*
- $|f'(0)| \leq 1$.

Equality holds for either of these conclusions if and only if $f(z) = e^{i\theta} z$ for some real θ.

Lemma 2.4 (Schwarz-Pick Lemma)**.** *If f is an analytic self-map of the unit disk $\mathbb{D}$, then for any pair of points z and w in the disk*

$$\rho(f(z), f(w)) \leq \rho(z, w).$$

Equality holds if and only if f is an automorphism, i.e $f(z) = e^{i\theta}\phi_a(z)$ for some $a \in \mathbb{D}$ and real θ.

By switching to the hyperbolic metric, G. Pick was able to remove the condition that $f(0) = 0$. This simplification reinforces the intuition that the hyperbolic disk assigns no special significance to the Euclidean origin at 0. This understand clashes dramatically with the conditions of the next lemma. Before stating this lemma, we define a "hyperbolic derivative".

Definition 2.5. A self-map f's **hyperbolic derivative** f^* is defined as
$$f^*(z) = f'(z)\frac{1-|z|^2}{1-|f(z)|^2}.$$

f^* is a self-map for non-automorphisms but it is not analytic. In fact, when f is an automorphism f^* maps $\mathbb{D}$ onto a portion of its boundary. This may be to a single point ($f(z) = z$) or to a section of the circle ($f(z) = \phi_{1/2}(z)$).

Along with the (first) hyperbolic derivative, in the last few years, higher hyperbolic derivatives $f^{[n]}$ have been defined in relation to standard higher derivatives $f^{(n)}$. Specifically, the following quantity is studied.
$$f^{[n]}(z) = f^{(n)}(z)\frac{(1-|z|^2)^n}{1-|f(z)|^2}$$

With the definitions and context provided above, we are ready to state the following lemma due to A. Beardon [2].

Lemma 2.6 (Beardon)**.** *If f is an analytic self-map of the unit disk $\mathbb{D}$, but not an automorphism, and if $f(0) = 0$, then*
$$\rho(f^*(0), f^*(z)) \leq 2\rho(0, z).$$
Equality holds when $f(z) = z^2$.

Remarkably, $f(0) = 0$ is an explicit condition and *cannot be removed easily*! This led to a series of attempts to remove it from researchers such as S. Yamashita [8], H. Kaptanoğlu [3] and P. Mercer [4]. An undergraduate could pursue the following program aimed at generalizing Beardon's result. Open questions are indicated with open bullets.

- • Write a function in Matlab or Maple to compute the hyperbolic distance between a pair of points.
- • Develop a method to generate random pairs of complex numbers that are uniformly distributed in $\mathbb{D}$.
- • Calculate f^* for simple functions including powers of z and low order[1] Blaschke products.
- • For $f(z) = z^2$, consider the quotient $\frac{\rho(f^*(a), f^*(z))}{\rho(a,z)}$ for fixed $a \in \mathbb{D}$ and variable z. Numerically estimate the maximum.
- • Previous simulations indicate that this value is $\frac{2(1+|a|^2)}{1-|a|^2}$. Determine where the maximum is attained? Use basic calculus (and possibly Maple) to derive the formula shown.
- ◦ Investigate the same quotient for other self-maps f where $f(a) = a^2$?
- ◦ Find a general formula for self-maps where $f(a) = b$ for a fixed pair of values $a, b \in \mathbb{D}$.
- • For simple examples of f, calculate the higher hyperbolic derivatives $f^{[n]}$ for $n = 2$ and 3.
- ◦ Develop a numerical method to calculate the quotient for these derivatives $\frac{\rho(f^{[n]}(a), f^{[n]}(z))}{\rho(a,z)}$. Does this quotient make sense? Why or why not? How might one modify $f^{[n]}$ to accomplish this goal?
- ◦ Assuming that a (modified) quotient is now available, maximize it numerically with an eye to the same issues raised previously.

3. Schwarz-Pick Lemmas For Schur Coefficients

Another sequence of values called "Schur coefficients", which are similar to hyperbolic derivatives, are worth investigating. These values were first analyzed by I. Schur [6, 7] and bear his name. A generalization[2] of their construction is described below. It is clear that $\gamma_1(a)$ is identical to $f^*(a)$.

---

[1] up to fourth

[2] The generalization was developed by the author but probably exists in the literature as well.

Definition 3.1. For an analytic self-map f and a fixed $a \in \mathbb{D}$, let $f_0 = f$ and $\gamma_0(a) = f(a)$. Define f_{n+1} and $\gamma_{n+1}(a)$ by iteration.

$$\begin{aligned} f_{n+1}(z) &= \frac{\phi_{\gamma_n(a)}(f_n(z))}{\phi_a(z)} \\ \gamma_{n+1}(a) &= f_{n+1}(a) = f_n'(a)\frac{1-|a|^2}{1-|f_n(a)|^2} \end{aligned}$$

Here would be the essential research program for an undergraduate with the same notational conventions as before.

- Each generalized Schur coefficient $\gamma_n(a)$ can be described in terms of $f(a)$ and the first n derivatives at a. Determine this formula for $n = 1, 2$ and 3.
- ∘ Fix a simple function[3] f and $n \geq 2$ and numerically maximize the quotient $\frac{\rho(\gamma_n(a),\gamma_n(b))}{\rho(a,b)}$. Start with $b = 0$.
- ∘ Use basic (differential) calculus and Maple to explain these maxima.

4. Bieberbach-Eilenberg Functions & The Schwarz-Christoffel Toolbox

Bieberbach-Eilenberg functions are generalizations of self-maps as will soon be evident.

Definition 4.1. A Bieberbach-Eilenberg function f is analytic in the unit disk $\mathbb{D}$ and satisfies the conditions that $f(0) = 0$ and $f(z)f(w) \neq 1$ for all $z, w \in \mathbb{D}$.

Some results regarding these functions are listed below. If they are not easily proven, they are listed with a ∘.

- $f(z) \neq \pm 1$ for all $z \in \mathbb{D}$.
- All self-maps of the unit disk fixing the origin ($f(0) = 0$) are Bieberbach-Eilenberg functions. Self-maps also satisfy the L^2 norm condition $\sum_{n=1}^{\infty} |a_k|^2 \leq 1$.
- ∘ Choose any simply connected region Ω that includes 0 and is bounded by a closed curve $\partial\Omega$ that satisfies the "self-reciprocal" condition $\partial\Omega = 1/\partial\Omega$, i.e. $\{w : w \in \partial\Omega\} = \{1/w : w \in \partial\Omega\}$. The Riemann mapping from $\mathbb{D}$ to Ω satisfying $f(0) = 0$ is automatically a Bieberbach-Eilenberg function.
  - Many domains Ω can be constructed out of portions of radial lines and semicircles centered at the origin. These are called "gear-like" regions.
  - Riemann mappings to these domains can be approximated using a *Schwarz-Christoffel Toolbox* for Matlab written by T. Driscoll. He has written modified m-files for gear-like domains.
- ∘ All Bieberbach-Eilenberg functions are bounded. The set of functions is neither uniformly bounded, convex nor is it star-like with respect to the origin. There are both even and odd Bieberbach-Eilenberg functions that are not self-maps of the unit disk.

While researchers worked to prove inequalities regarding each independent coefficient of a Bieberbach-Eilenberg function, Z. Nehari [5] and D. Aharonov [1] made the following remarkable discovery regarding the L^2 norm.

Theorem 4.2. *If f is a Bieberbach-Eilenberg function, then* $\sum_{k=1}^{\infty} |a_k|^2 \leq 1$.

An undergraduate possesses the tools to determine whether there are Bieberbach-Eilenberg functions that are not self-maps which are "maximal" in the L^2 sense. Many self-maps satisfy the condition that $\sum_{k=1}^{\infty} |a_k|^2 = 1$. What about the aforementioned Riemann maps? The following tasks would further prepare an undergraduate to address this question.

- Explain why a Bieberbach-Eilenberg function must be bounded. (*Hint*: $f(0) = 0$.)

[3] The order of z^m or the Blaschke product must be greater than n

- Given f, how could one test if it was a Bieberbach-Eilenberg function graphically? (*Hint*: Try drawing its boundary.)
- ◦ Find quadratic and cubic polynomials that are Bieberbach-Eilenberg functions but are not self-maps.[4]
- How is the integral[5] $\int_0^{2\pi} |f(e^{it})|^2 \, dt$ related to $\sum_{k=1}^{\infty} |a_k|^2$?
- ◦ What error bounds does one encounter when performing any numeric integration? What inherent errors accompany the Schwarz-Christoffel Toolbox?
- ◦ Use Matlab and numeric integration to check whether a large class of Riemann maps are maximal with respect to the L^2 norm.

5. Rearrangements of Analytic Self-Maps

Unlike Bieberbach-Eilenberg functions, analytic self-maps form a convex set. This convexity allows for a nice decomposition of an analytic self-map into constituent self-maps using a primitive nth root of unity $\zeta = e^{2\pi i/n}$. For example, $f_{odd}(z) = \frac{f(z)-f(-z)}{2}$ and $f_{even}(z) = \frac{f(z)+f(-z)}{2}$ are both self-maps as well. More general decompositions can be defined.

Definition 5.1. Let f be an analytic in $\mathbb{D}$, fix an integer value $n \geq 2$, an integer value $0 \leq j \leq n-1$, and ζ a primitive root of unity. The function f's (n, j)-th part $f_{n,j}$ and its related function $\hat{f}_{n,j}$ are defined as

$$f_{n,j}(z) = \frac{1}{n} \sum_{k=0}^{n-1} w^{kj} f(\zeta^k z) = \sum_{k=0}^{\infty} a_{nk+j} z^{nk+j} = z^j \hat{f}_{n,j}(z^n).$$

Each of these functions are self-maps. As expected, $f_{2,0} = f_{even}$ and $f_{2,1} = f_{odd}$. Of course, the odd and even parts of f are more than independent self-maps. They combine to form a unified self-map as well.

However, what if these pieces were reassembled in a new way? For example $g(z) = z f_{even}(z) + \frac{1}{z} f_{odd}(z)$. How does this reassembly affect the function's norms? Clearly, the L^2 norm remains unchanged, but what about the supremum norm? What is the largest that g can be? To address this question, we introduce a more general notion of a rearranged function.

Definition 5.2. For a given self-map f and fixed integer $n \geq 2$, choose a permutation σ of the integers from 0 to $n-1$. The rearranged function f_σ is given by the formula

$$\sum_{j=0}^{n-1} z^{\sigma(j)} \hat{f}_{n,j}(z^n).$$

An undergraduate project to examine the modified supremum begins by fixing an arbitrary z and identifying each image as a value $w_j = f_{n,j}(z)$. Since f is a self-map, a series of n inequalities emerges relating the w_j's. Moreover, any rearrangement f_σ would be limited by the inequality $|f_\sigma(z)| \leq |w_0| + |w_1| + \cdots + |w_{n-1}|$.

- For values of $n \leq 10$, formulate these inequalities. For example, for $n = 2$, the inequalities are $|w_0 + w_1| \leq 1$ and $|w_0 - w_1| \leq 1$.
- Generalize the **parallelogram law**: $|w_0 + w_1|^2 + |w_0 - w_1|^2 = 2(|w_0|^2 + |w_1|^2)$ to values of $n > 2$. (*Hint*: Use ζ.)
- For positive real values $x_0, x_1, \ldots, x_{n-1}$ satisfying the condition $x_0^2 + x_1^2 + \ldots + x_{n-1}^2 = 1$, locate the maximum of $x_0 + x_1 + \cdots + x_{n-1}$ and determine its value.

---

[4] We are unfamiliar with such results, but suspect that they are known.

[5] For simplicity, we assume that f is well-defined on the boundary.

- Randomly generate n complex numbers in $\{w_j\} \in \mathbb{D}$, $0 \leq j \leq n-1$ that satisfy the conditions listed above and maximize the sum of their moduli $\sum_{j=0}^{n-1} |w_j|$. For which values of w_j does this maximum occur? Is the maximum for the complex values w_j the same as the maximum for the real values x_j?
- If the complex maximum is smaller, describe and explain it.
- Find a self-map f with modulus 1 and a rearrangement f_σ that attains this maximum modulus.

References

[1] Dov Aharonov. On bieberbach eilenberg functions. *Bulletin of the AMS*, 76:101–104, 1970.

[2] A. F. Beardon. The schwarz-pick lemma for derivatives. *Proceedings of the American Mathematical Society*, 125(11):3255–3256, 1997.

[3] H. Turgay Kaptanoğlu. Some refined schwarz-pick lemmas. *The Michigan Mathematical Journal*, 3:649–664, 2002.

[4] Peter R. Mercer. Schwarz-pick-type estimates for the hyperbolic derivative. *Journal of Inequalities and Applications*, 2006:Article ID 96368, 6 pages, 2006. doi:10.1155/JIA/2006/96368.

[5] Zeev Nehari. On the coefficients of bieberbach-eilenberg functions. *Journal d'Analyse Mathématique*, 23:297–303, 1970.

[6] I. Schur. Über potenzreihen die im inneren des einheitskreises beschränkt sind i. *Journal Reine Angew. Math.*, 147:205–232, 1917.

[7] I. Schur. Über potenzreihen die im inneren des einheitskreises beschränkt sind ii. *Journal Reine Angew. Math.*, 148:122–145, 1918.

[8] Shinji Yamashita. The pick version of the schwarz lemma and comparison of the poincaré densities. *Annales AcademicæScientiarum Fennicæ*, 19:291–322, 1994.