Examples of Bieberbach-Eilenberg (BE) Functions

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Introduction & Definition

- f Analytic In $D = \{|z| < 1\}$, $f(0) = 0$
- $\boxed{f(z)f(w) \neq 1 \text{ For All } z, w \in D.}$
- $f(z) = \sum_{n=1}^{\infty} a_n z^n$.
- Generalization of Self-maps:
  - $\star$ $f : D \to D$, i.e. $\|f\|_\infty \leq 1$, $f(0) = 0$
  - $\star$ Generally, $f = zg$, $g : D \to D$.
- Can't Remove 0 Easily!

History - Coefficient Inequalities

- **1970** D. Aharonov, Z. Nehari:
  $\|f\|_2^2 = \sum_{n=1}^{\infty} |a_n|^2 \leq 1$

- BE Functions Inside Convex Ball.

- Schur Inequalities (For Self-Maps)

  $\star$ $|a_n| \leq 1 - |a_k|^2$, $k < \frac{n}{2}$.

  $\star$ e.g., $|a_3| \leq 1 - |a_1|^2$

  $\star$ e.g., $|a_2| \leq 1 - |a_1|^2$ (if $f(0) = 0$).

  $\star$ False For BE and $f(z) = \frac{z - \frac{1}{3}}{1 - \frac{1}{3}z}$.

Geometric Questions

- Is the Set of BE Functions:
  - ⋆ Bounded? (Yes/No)
  - ⋆ Convex? (No)
  - ⋆ **Starlike? (No)**
- Who Cares?
  - ⋆ If f is BE, is f_{odd} or f_{even} BE?
  - ⋆ Are There Any Odd BE Functions?

Transformation & Subordination

- Transformations for Self-Maps:
  - $\star$ $f \to \frac{f(z) \pm f(-z)}{2}$, $\frac{1}{z}\frac{f(z)-f(0)}{1-\overline{f(0)}f(z)}$
- For BE Functions
  - $\star$ $\phi : D \to D$, $\phi(0) = 0$.
  - $\star$ $F \in BE \Rightarrow f(z) = \pm F(\phi(z)) \in BE$.
- Subordination: $f \prec F$ (Rogosinski)
  - $\star$ f BE $\Rightarrow f \prec F$ BE **Univalent**.
  - $\star$ HOW CAN THIS BE?!?

Riemann Mapping Theorem

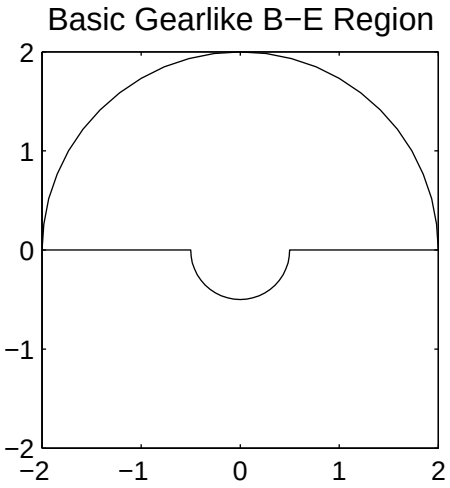
- $f(D)$ In Smallest Simply Connected Region Ω.
- $F : D \to \Omega$ Univalent, $F(0) = 0$.
- Why Not Generate BE's Geometrically?
- **Alternative Definition**: $\{w, \frac{1}{w}\} \notin f(D)$.
- Pick Simple Closed Curve γ Enclosing Ω:
  - $\star$ $\frac{1}{\gamma} = \gamma$, $0 \in \Omega$
- Let $f : D \to \Omega$, $f(0) = 0$ BE Univalent.
- Constructs **Extremal** BE Domains (Functions)

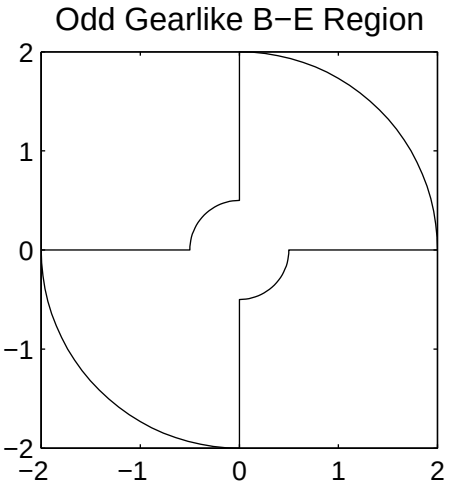
Goals: Compute & Conjecture

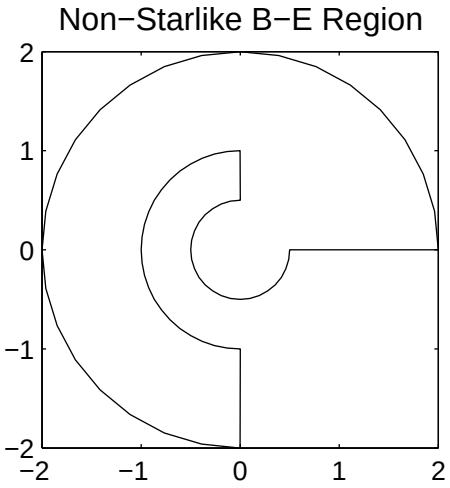
- Conjectures:
  - ⋆ f BE Extremal $\Rightarrow \sum |a_n|^2 = 1$.
  - ⋆ Test a Schur inequality ($|a_3| \leq 1 - |a_1|^2$).
- Computation (SC Toolbox)
  - ⋆ Polygons, Gearlike, Circular Arcs.
  - ⋆ Shifted Circle ($r = 2$): $\sum |a_n|^2 = 1$.
  - ⋆ Simple Gearlike ($r = 2$): $\sum |a_n|^2 \approx 0.8876 < 1$
  - ⋆ Odd Gearlike ($r = 2$): $\sum |a_n|^2 \approx 0.6706 < 1$

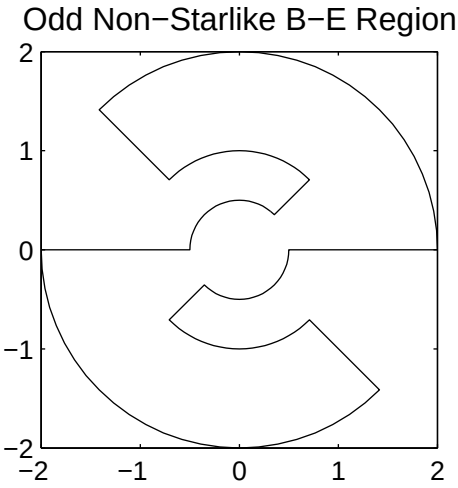
First Coefficients

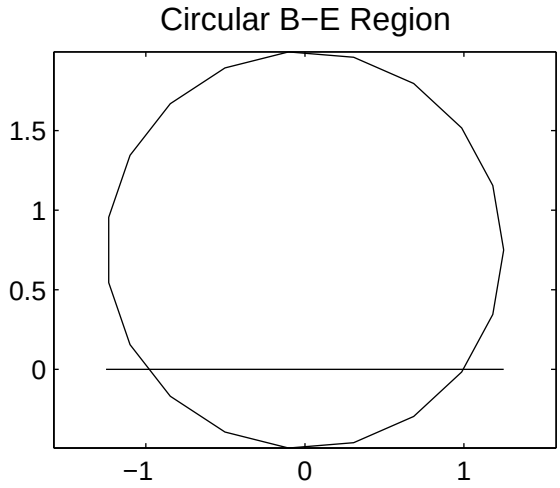
Region	$\lvert a_1 \rvert$	$\lvert a_2 \rvert$	$\lvert a_3 \rvert$	$\lvert a_4 \rvert$
Circle	.8	$.8(.6)$	$.8(.6)^2$	$.8(.6)^3$
Gear				
Odd Gear				

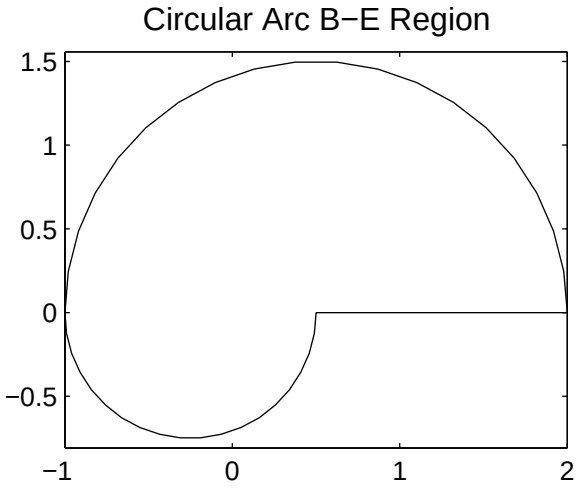

Basic Gearlike B−E Region
2
1
0
−1
−2
−2
−1
0
1
2

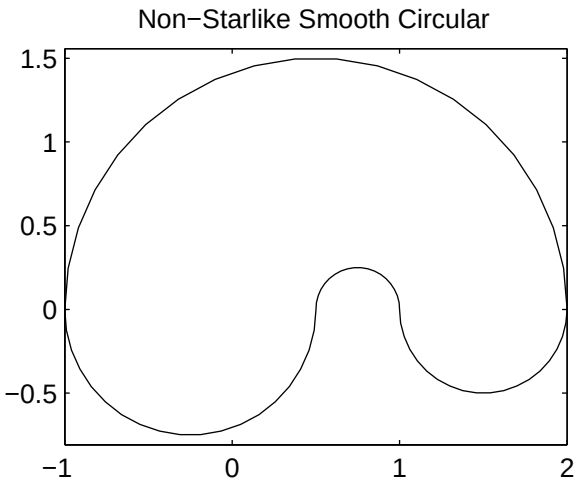

Odd Gearlike B−E Region
2
1
0
−1
−2
−2
−1
0
1
2

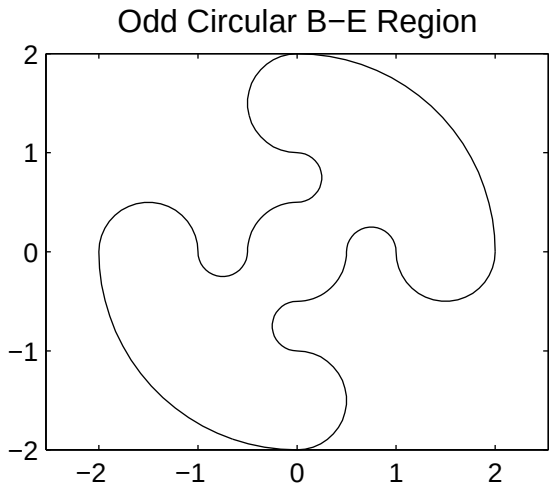

Non−Starlike B−E Region
2
1
0
−1
−2
−2
−1
0
1
2


Odd Non−Starlike B−E Region
2
1
0
−1
−2
−2
−1
0
1
2


Circular B−E Region
1.5
1
0.5
0
−1
0
1


Circular Arc B−E Region
1.5
1
0.5
0
−0.5
−1
0
1
2


Non−Starlike Smooth Circular
1.5
1
0.5
0
−0.5
−1
0
1
2


Odd Circular B−E Region
2
1
0
−1
−2
−2
−1
0
1
2