Lecture Notes on Minimal Surfaces

January 27, 2006

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1 Some Background in Differential Geometry

Our goal is to develop the mathematics necessary to investigate minimal surfaces in $\mathbb{R}^3$. Every point on a surface $S \subset \mathbb{R}^3$ can be designated by a point, $(x^1, x^2, x^3) \in \mathbb{R}^3$, but it can also be represented by two parameters. Let D be an open set in $\mathbb{R}^2$. Then a surface, $S \in \mathbb{R}^3$, can be represented by a function $\mathbf{x} : D \to \mathbb{R}^3$, where $\mathbf{x}(u, v) = (x^1, x^2, x^3)$. Such a function or mapping is called a *parametrization*.

Example 1.

1. For the torus, let

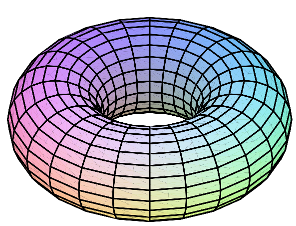
$$\mathbf{x}(u, v) = \Big((a + b\cos v)\cos u, (a + b\cos v)\sin u, b\sin v\Big),$$

   where a, b are fixed, $0 < u < 2\pi$, and $0 < v < 2\pi$.

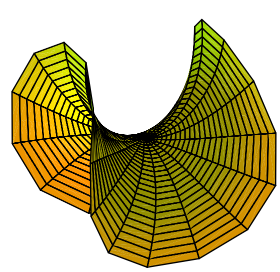
2. For Enneper's surface, let

$$\mathbf{x}(u, v) = \left(u - \frac{1}{3}u^3 + uv^2, v - \frac{1}{3}v^3 + vu^2, u^2 - v^2\right),$$

   where u, v are in a disk of radius r.



Torus



Enneper's surface

Suppose $\mathbf{x}(u,v)$ is a parametrization of a surface $S \in \mathbb{R}^3$. If we fix $v = v_0$ and let u vary, then $\mathbf{x}(u, v_0)$ depends on one parameter and is hence a curve called the *u-parameter curve*. Likewise, we can fix $u = u_0$ and let v vary to get the *v-parameter curve* $\mathbf{x}(u_0, v)$. Tangent vectors for the u-parameter and v-parameter curves are computed by differentiating the component functions of $\mathbf{x}$ with respect to u and v, respectively. That is,

$$\mathbf{x}_u = \left(\frac{\partial x^1}{\partial u}, \frac{\partial x^2}{\partial u}, \frac{\partial x^3}{\partial u}\right), \qquad \mathbf{x}_v = \left(\frac{\partial x^1}{\partial v}, \frac{\partial x^2}{\partial v}, \frac{\partial x^3}{\partial v}\right).$$

Example 2. A torus can be parametrized by

$$\mathbf{x}(u,v) = \Big((3 + 2\cos v)\cos u, (3 + 2\cos v)\sin u, 2\sin v\Big),$$

where $0 < u, v < 2\pi$. For $v_0 = \frac{\pi}{2}$, the u-parameter curve is

$$\mathbf{x}\left(u, \frac{\pi}{2}\right) = (0, 3 + 2\cos u, 2\sin u).$$

For $u_0 = \frac{\pi}{3}$, the v-parameter curve is

$$\mathbf{x}\left(\frac{\pi}{3}, v\right) = (4\cos v, 2\sin v, \sqrt{3}).$$

Notice

$$\begin{aligned}\mathbf{x}_u(u,v) &= (-2\sin u\cos v, -2\sin u\sin v, 2\cos u)\\ \mathbf{x}_v(u,v) &= (-(3 + 2\cos u)\sin v, (3 + 2\cos u)\cos v, 0).\end{aligned}$$

So

$$\begin{aligned}\mathbf{x}_u\left(\frac{\pi}{3}, \frac{\pi}{2}\right) &= (0, -\sqrt{3}, 1)\\ \mathbf{x}_v\left(\frac{\pi}{3}, \frac{\pi}{2}\right) &= (-4, 0, 0).\end{aligned}$$

What do these quantities represent? The tangent vector to the u- and v-parameter curves at the point $\mathbf{x}(\frac{\pi}{3}, \frac{\pi}{2})$.

Whenever we have a parametrization of a surface, we will require that $\mathbf{x}_u$ and $\mathbf{x}_v$ be linearly independent. Because of this, the span of $\mathbf{x}_u$ and $\mathbf{x}_v$ (i.e., the set of all vectors that can be written as a linear combination of $\mathbf{x}_u$, $\mathbf{x}_v$) forms a plane called the *tangent plane*.

Definition 3. The *tangent plane* of a surface M at a point p is

$$T_pM = \{\mathbf{v} \big| \mathbf{v} \text{ is tangent to } M \text{ at } p\}.$$

Definition 4. The *unit normal* to a surface M at a point $p = \mathbf{x}(a, b)$ is

$$\mathbf{n}(a,b) = \left.\frac{\mathbf{x}_u \times \mathbf{x}_v}{\left|\mathbf{x}_u \times \mathbf{x}_v\right|}\right|_{(a,b)}.$$

Note that the unit normal, $\mathbf{n}$, is perpendicular to the tangent plane at p. Also, if the surface M is oriented, then there are two unit normals at each point $p \in M$–an outward pointing normal and an inward pointing normal. Any plane that contains this normal $\mathbf{n}$ will intersect the surface M in a curve, $\mathbf{c}$. For each curve $\mathbf{c}$, we can compute it curvature which measures how fast the curve pulls away from the tangent line at p. So let's now review some ideas about the curvature of a line.

Any curve in $\mathbb{R}^3$ can be parametrized by one variable, say $\mathbf{c}(t)$, where $\mathbf{c} : (a, b) \to \mathbb{R}^3$. For example, $\mathbf{c}(t) = (\cos t, \sin t, t)$ for $t \in \mathbb{R}$ is a parametrization of a helix.

c '

Definition 5. A curve $\mathbf{c}$ is a *unit speed curve* if $\left|\mathbf{c}'(t)\right| = 1$.

Any nonunit speed regular curve $\mathbf{c}$ can be reparametrized by arclength to form a unit speed curve $\mathbf{c}(s)$. So we will assume that the curves we will be discussing are unit speed curves $\mathbf{c}(s)$. This assumption means that we are only interested in the geometric shape of a regular curve since reparametrizing does not change the shape of a curve.

Given a curve $\mathbf{c}$, we want to discuss its curvature (or bending). The amount of bending of the curve is demonstrated by the measure of how rapidly the curve pulls away from the tangent line at p. In other words, it measures the rate of change of the angle θ that neighboring tangents make with the tangent at p. Thus, we are interested in the rate of change of the tangent vector (i.e., the value of the second derivative).

Definition 6. The *curvature* of $\mathbf{c}$ at s is $\left|\mathbf{c}''(s)\right|$.

Example 7. What is the curvature of a circle at point p?

The location of p does not matter, because of the symmetry of the circle. Recall that a circle of radius r can be parametrized by

$$\mathbf{c}(t) = (r\cos(bt), r\sin(bt), 0).$$

Notice that this is not a unit speed curve since

$$\begin{aligned}\mathbf{c}'(t) &= (-rb\sin(bt), -rb\cos(bt), 0)\\ &\Rightarrow 1 = \left|\mathbf{c}'(t)\right| = rb\\ &\Rightarrow b = \frac{1}{r}.\end{aligned}$$

So we reparametrized it into a unit speed curve by letting $\dfrac{s}{r} = bt$ to get

$$\mathbf{c}(s) = \left(r\cos\frac{s}{r}, r\sin\frac{s}{r}, 0\right).$$

Then

$$\begin{aligned}
\mathbf{c}'(s) &= \left(-\sin\frac{s}{r}, \cos\frac{s}{r}, 0\right) \\
\mathbf{c}''(s) &= \left(-\frac{1}{r}\cos\frac{s}{r}, -\frac{1}{r}\sin\frac{s}{r}, 0\right) \\
\left|\mathbf{c}''(s)\right| &= \frac{1}{r}.
\end{aligned}$$

That is, the curvature of a circle of radius r is $\dfrac{1}{r}$. Notice that if r is large then the curvature is small, while if r is small the curvature is large.

Now let's return to surfaces. Suppose we have a curve $\sigma(s)$ on a surface M. We can determine the unit tangent vector, $\mathbf{w}$ of σ at $p \in M$ and the unit normal, $\mathbf{n}$ of M at $p \in M$. Note that $\mathbf{w} \times \mathbf{n}$ forms a plane $\mathcal{P}$ that intersects M creating a curve $\mathbf{c}(s)$.

Definition 8. The *normal curvature* in the $\mathbf{w}$ direction is

$$k(\mathbf{w}) = \mathbf{c}'' \cdot \mathbf{n}.$$

Recall $\mathbf{c}'' \cdot \mathbf{n} = \left|\mathbf{c}''\right|\left|\mathbf{n}\right|\cos\theta$. Hence $\mathbf{c}'' \cdot \mathbf{n}$ is the projection of $\mathbf{c}''$ onto the unit normal (hence, the name *normal* curvature).

As we rotate the plane through the normal $\mathbf{n}$, we will get a set of curves on the surface each of which has a value for its curvature. Let k_1 and k_2 be the maximum and minimum curvature values at p, respectively.

Definition 9. The *mean curvature* (i.e., average curvature) of a surface M at p is

$$H = \frac{k_1 + k_2}{2}.$$

It turns out that k_1 and k_2 come from two perpendicular tangent vectors. The mean curvature depends upon the point $p \in M$. However, it can be shown that H does not change if we choose any two perpendicular vectors and use their curvature values to compute H at p.

Examples

1. At any point on a sphere of radius a, all the curves $\mathbf{c}$ are circles of radius a and hence have the same curvature value which can be computed to be $1/a$. So the mean curvature is $1/a$.

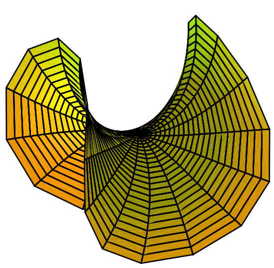
2. For Enneper's surface, the values for k_1 and k_2 vary at each point. However, $k_2 = -k_1$ at each point and so the mean curvature equals 0 at each point. In such a case, the surface is called a minimal surface.

2 Minimal Surfaces

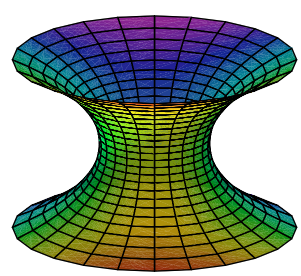
2.1 Basics

Definition 10. A *minimal surface* is a surface M with $H = 0$ at all $p \in M$.

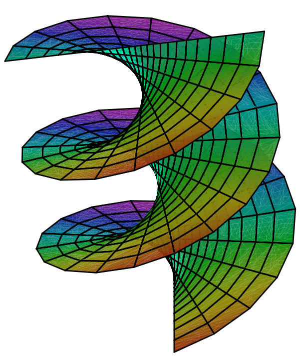
Example 11. Some standard examples of minimal surfaces in $\mathbb{R}^3$ are: the plane, Enneper's surface, the catenoid, the helicoid, and Scherks doubly-periodic surface. Pictures of some standard minimal surfaces:



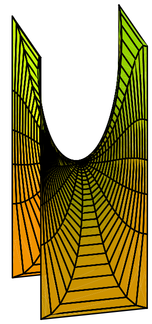
Enneper's surface



catenoid



helicoid



Scherks doubly-periodic

Definition 10 is not practical for determining if a surface is minimal. However, there is a nice formula using the coefficients of the first and second fundamental forms for a surface.

Recall that $\mathbf{c}$ is a unit speed curve. Hence

$$\begin{aligned} 1 =& |\mathbf{c}'|^2 = \mathbf{c}' \cdot \mathbf{c}' \\ =& (\mathbf{x}_u \, du + \mathbf{x}_v \, dv) \cdot (\mathbf{x}_u \, du + \mathbf{x}_v \, dv) \\ =& \mathbf{x}_u \cdot \mathbf{x}_u \, du^2 + 2\mathbf{x}_u \cdot \mathbf{x}_v \, dudv + \mathbf{x}_v \cdot \mathbf{x}_v \, dv^2 \\ =& E \, du^2 + 2F \, dudv + G \, dv^2. \end{aligned} \tag{1}$$

The terms $E = \mathbf{x}_u \cdot \mathbf{x}_u$, $F = \mathbf{x}_u \cdot \mathbf{x}_v$, and $G = \mathbf{x}_v \cdot \mathbf{x}_v$ are known as the *coefficients of the first fundamental form*. This describes how lengths on a surface are distorted as compared to their usual measurements in $\mathbb{R}^3$.

Next, recall $k(w) = \mathbf{c}'' \cdot \mathbf{n}$. Note that $\mathbf{c}'' \cdot \mathbf{n} = -\mathbf{c}' \cdot \mathbf{n}'$, because $\mathbf{c}' \cdot \mathbf{n} = 0$ and so $(\mathbf{c}' \cdot \mathbf{n})' = 0$ which implies $\mathbf{c}'' \cdot \mathbf{n} + \mathbf{c}' \cdot \mathbf{n}' = 0$. Similarly, $-\mathbf{x}_u \cdot \mathbf{n}_u = \mathbf{x}_{uu} \cdot \mathbf{n}$. So

$$\begin{aligned} k(w) =& - \mathbf{c}' \cdot \mathbf{n}' \\ =& - (\mathbf{x}_u \, du + \mathbf{x}_v \, dv) \cdot (\mathbf{n}_u \, du + \mathbf{n}_v \, dv) \\ =& - \mathbf{x}_u \cdot \mathbf{n}_u \, du^2 - (\mathbf{x}_u \cdot \mathbf{n}_v + \mathbf{x}_v \cdot \mathbf{n}_u) \, dudv - \mathbf{x}_v \cdot \mathbf{n}_v \, dv^2 \\ =& \mathbf{x}_{uu} \cdot \mathbf{n} \, du^2 + 2\mathbf{x}_{uv} \cdot \mathbf{n} \, dudv + \mathbf{x}_{vv} \cdot \mathbf{n} \, dv^2 \\ =& e \, du^2 + 2f \, dudv + g \, dv^2. \end{aligned}$$

The terms $e = \mathbf{x}_{uu} \cdot \mathbf{n}$, $f = \mathbf{x}_{uv} \cdot \mathbf{n}$, and $g = \mathbf{x}_{vv} \cdot \mathbf{n}$ are called *the coefficients of the second fundamental form*.

Now we want to express the mean curvature H in terms of these coefficients of the first and second fundamental forms. In particular, we will show that

$$H = \frac{Eg + Ge - 2Ff}{2(EG - F^2)}. \tag{2}$$

Proof. Let $\mathbf{w}_1, \mathbf{w}_2$ be any two perpendicular unit vectors. Let k_1, k_2 be their normal curvatues using the curves $\mathbf{c}_1$, $\mathbf{c}_2$ with parameters $u_1(s), v_1(s)$ and $u_2(s), v_2(s)$. Let's denote $p_1 = du_1 + idu_2$ and $p_2 = dv_1 + idv_2$. Then

$$\begin{aligned} 2H = k_1 + k_2 &= e(du_1^2 + du_2^2) + 2f(du_1dv_1 + du_2dv_2) + g(dv_1^2 + dv_2^2) \\ &= e(p_1\,\overline{p_1}) + f(p_1\,\overline{p_2} + \,\overline{p_1}p_2) + g(p_2\,\overline{p_2}). \end{aligned}$$

We want to further simplify this so that it does not have p_1 and p_2. Recall eq (1):

$$1 = E\,du^2 + 2F\,dudv + G\,dv^2.$$

Now consider

$$\begin{aligned} Ep_1^2 + 2Fp_1p_2 + Gp_2^2 &= E\big[du_1^2 - du_2^2 + i2du_1du_2\big] + 2F\big[du_1dv_1 - du_2dv_2 + i(du_1dv_2 + du_2dv_1)\big] \\ &\qquad + G\big[dv_1^2 - dv_2^2 + i2dv_1dv_2\big] \\ &= 2i\big[Edu_1du_2 + F(du_1dv_2 + du_2dv_1) + Gdv_1dv_2\big] \\ &\qquad + \big[Edu_1^2 + 2Fdu_1dv_1 + Gdv_1^2\big] - \big[Edu_2^2 + 2Fdu_1dv_2 + Gdv_2^2\big] \\ &= 0 + 1 - 1 \\ &= 0. \end{aligned}$$

Thus,

$$\begin{aligned} p_1 &= \frac{-2Fp_2 \pm \sqrt{4F^2p_2^2 - 4EGP_2^2}}{2E} = \left(-\frac{F}{E} \pm i\frac{\sqrt{EG - F^2}}{E} \right) p_2 \\ \overline{p_1} &= \left(-\frac{F}{E} \mp i\frac{\sqrt{EG - F^2}}{E} \right) \overline{p_2}. \end{aligned}$$

And so

$$p_1\,\overline{p_1} = \left(\frac{F^2}{E^2} + EG - F^2E \right) p_2\,\overline{p_2} = \frac{G}{E} p_2\,\overline{p_2} \tag{3}$$

$$p_1\,\overline{p_2} + \,\overline{p_1}p_2 = -\frac{2F}{E} p_2\,\overline{p_2}. \tag{4}$$

Now we have

$$\begin{aligned} 2H = k_1 + k_2 &= e(p_1\,\overline{p_1}) + f(p_1\,\overline{p_2} + \,\overline{p_1}p_2) + g(p_2\,\overline{p_2}) \\ &= \left[e\frac{G}{E} + f\left(\frac{-2F}{E} \right) + g \right] p_2\,\overline{p_2}. \end{aligned}$$

We just need to get rid of $p_2\,\overline{p_2}$. Again using eq (1), we have

$$\begin{aligned} &Ep_1\,\overline{p_1} + F(p_1\,\overline{p_2} + \,\overline{p_1}p_2) + Gp_2\,\overline{p_2} \\ &\quad = E(du_1^2 + du_2^2) + 2F(du_1dv_1 + du_2dv_2) + G(dv_1^2 + dv_2^2) \\ &\quad = 1 + 1 = 2. \end{aligned}$$

Using eqs (3) and (4),we derive

$$\begin{aligned}
2 &= E\left(\frac{G}{E}p_2\,\overline{p_2}\right) + 2F\left(\frac{-2F}{E}p_2\,\overline{p_2}\right) + Gp_2\,\overline{p_2} \\
\Rightarrow 2 &= \left[2G - \frac{2F^2}{E}\right]p_2\,\overline{p_2} \\
\Rightarrow p_2\,\overline{p_2} &= \frac{E}{EG - F^2}
\end{aligned}$$

Therefore,

$$H = \frac{1}{2}\left[e\frac{G}{E} + f\left(\frac{-2F}{E}\right) + g\right]p_2\,\overline{p_2} = \frac{Eg + eG - 2Ff}{2(EG - F^2)}.$$

□

We can use formula (2) to show that a surface with a specific parametrization is minimal.

Example 12. If a heavy flexible cable is suspended between two points at the same height, then it takes the shape of a curve $x = a\cosh(z/a)$ called a catenary (from the Latin word that means "chain"). A *catenoid* which is a surface generated by rotating this catenary about the z-axis. How do we parametrize this catenary? Let $z = av$. Then $x = a\cosh v(-\infty < v < \infty)$. So $r(v) = (a\cosh v, av)$ in the xz-plane. Since the catenoid is formed by rotating this curve, we get the following parametrization for a catenoid:

$$\mathbf{x}(u,v) = (a\cosh v\cos u, a\cosh v\sin u, av).$$

Then

$$\begin{aligned}
\mathbf{x_u} &= (-a\cosh v\sin u, a\cosh v\cos u, 0) \\
\mathbf{x_v} &= (a\sinh v\cos u, a\sinh v\sin u, a).
\end{aligned}$$

So

$$\begin{aligned}
E =& \mathbf{x}_u \cdot \mathbf{x}\mathbf{x}_u = a^2\cosh^2 v \\
F =& \mathbf{x}_u \cdot \mathbf{x}_v = 0 \\
G =& \mathbf{x}_v \cdot \mathbf{x}_v = a^2\cosh^2 v
\end{aligned}$$

Next, we want to compute $n = \dfrac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}$. Now,

$$\mathbf{x}_u \times \mathbf{x}_v = (a^2\cosh v\cos u, a^2\cosh v\sin u, -a^2\cosh v\sinh v),$$

and so

$$|\mathbf{x}_u \times \mathbf{x}_v| = a^2\cosh^2 v.$$

Hence

$$\mathbf{n} = \left(\frac{\cos u}{\cosh v}, \frac{\sin u}{\cosh v}, -\frac{\sinh v}{\cosh v} \right).$$

and

$$\begin{aligned}
\mathbf{x}_{uu} =& (-a\cosh v \cos u, -a\cosh v \sin u, 0) \\
\mathbf{x}_{uv} =& (-a\sinh v \sin u, -a\sinh v \cos u, 0) \\
\mathbf{x}_{vv} =& (a\cosh v \cos u, a\cosh v \sin u, 0)
\end{aligned}$$

Then

$$\begin{aligned}
e =& \mathbf{n} \cdot \mathbf{x}_{uu} = -a \\
f =& \mathbf{n} \cdot \mathbf{x}_{uv} = 0 \\
g =& \mathbf{n} \cdot \mathbf{x}_{vv} = a,
\end{aligned}$$

which implies

$$H = \frac{1}{2} \frac{eG - 2fF + Eg}{EG - F^2} = 0.$$

And so the catenoid is a minimal surface.

Exercise 1. Prove that Enneper's surface parametrized by

$$\mathbf{x}(u, v) = \left(u - \frac{1}{3}u^3 + uv^2, v - \frac{1}{3}v^3 + vu^2, u^2 - v^2 \right).$$

is a minimal surface.

Exercise 2. Suppose a surface M is the graph of a function of two variables (i.e., if we project the surface onto a domain in the x_1x_2-plane, then for every point in that domain there is exactly one point of M over it). Then $x_3 = f(x_1, x_2)$ for some function f. M can then be parametrized by

$$\mathbf{x}(u, v) = \big(u, v, f(u, v)\big),$$

where $u \times v$ is the domain formed by the projection of M onto the x_1x_2-plane. Prove

$$M \text{ is a minimal graph } \iff f_{uu}\Big(1 + f_v^2\Big) - 2f_uf_vf_{uv} + f_{vv}\Big(1 + f_u^2\Big) = 0. \quad (5)$$

Applying eq (5) is usually not easy, because f can be complicated. However, one case in which we can solve for f is when f can be separated into two functions each of which is dependent upon only one variable. In particular, $f(x, y) = g(x) + h(y)$. Then the minimal surface equation becomes:

$$g''(x)[1 + (h'(y))^2] + h''(y)[1 + (g'(x))^2] = 0.$$

This is a separable differential equation and hence can be solved fairly easily. To do so, put all the x's on one side and all the y's on the other side to obtain:

$$-\frac{1 + (g'(x))^2}{g''(x)} = \frac{1 + (h'(y))^2}{h''(y)}. \tag{6}$$

What does this mean? If we keep y fixed, the right side remains the same as we vary x. Likewise, varying y has no effect on the left side of the equation. The only way that this can occur is if both sides equal the same constant, say c. So we have:

$$-\frac{1 + (g'(x))^2}{g''(x)} = c \qquad \Longrightarrow \qquad 1 + (g'(x))^2 = -cg''(x).$$

To solve this, let $\phi(x) = g'(x)$. Then $\frac{d\phi}{dx} = g''(x)$ and so

$$\begin{aligned} -\int dx &= -c \int \frac{d\phi}{1 + \phi^2} \\ \Longrightarrow \quad x &= -c \arctan \phi + K \\ \Longrightarrow \quad \phi &= -\tan\left(\frac{x + K}{c}\right). \end{aligned}$$

For convenience, let $K = 0$ and $c = 1$. Since $\phi = g'$, we can integrate again to get:

$$g(x) = \ln[\cos x].$$

Completing the same calculations for the y-side of eq(6) yields:

$$h(y) = -\ln[\cos y].$$

Hence

$$f(x, y) = g(x) + h(y) = \ln\left[\frac{\cos x}{\cos y}\right]$$

which is the equation for Scherks doubly periodic surface. Notice that $-\frac{\pi}{2} < x, y < \frac{\pi}{2}$ and so this surface is defined over a square with side lengths π centered at the origin. By a theorem known as the Schwarz Reflection Pronciple, we can fit pieces of Scherks doubly periodic surface together to get a checkerboard domain.

2.2 Isothermal parameters

By requiring the parametrization of a minimal surface to be isothermal, we can begin using complex analysis to help us better understand minimal surfaces.

Definition 13. A parametrization $\mathbf{x}$ is *isothermal* if $E = \mathbf{x}_u \cdot \mathbf{x}_u = \mathbf{x}_v \cdot \mathbf{x}_v = G$ and $F = \mathbf{x}_u \cdot \mathbf{x}_v = 0$.

This requirement is not a restriction because of the following theorem.

Theorem 14. Isothermal coordinates exist on any minimal surface in $\mathbb{R}^3$.

Proof. See Oprea, pp 73-75. □

Theorem 15. If the parametrization $\mathbf{x}$ is isothermal, then

$$\mathbf{x_{uu}} + \mathbf{x_{vv}} = 2EH\mathbf{n}.$$

Exercise 3 (A proof of Theorem 15). $\{\mathbf{x}_u, \mathbf{x}_v, \mathbf{n}\}$ forms a basis for $\mathbb{R}^3$. Assume $F = 0$. Then the vector $\mathbf{x}_{uu}$ can be expressed in terms of these bases vectors. That is,

$$\mathbf{x}_{uu} = \Gamma^u_{uu}\mathbf{x}_u + \Gamma^v_{uu}\mathbf{x}_v + e\mathbf{n},$$

where the Γ's coefficients are called the Christoffel symbols (recall $e = \mathbf{n} \cdot \mathbf{x}_{uu}$).

1. Show that $\Gamma^u_{uu} = \frac{E_v}{2E}$ and $\Gamma^v_{uu} = -\frac{E_v}{2G}$ by taking the inner product of $\mathbf{x}_{uu}$ with appropriate vectors. In a similar manner, it can be shown that

$$\mathbf{x}_{vv} = -\frac{G_u}{2E}\mathbf{x}_u + \frac{G_v}{2G}\mathbf{x}_v + g\mathbf{n}.$$

2. Use the mean curvature equation (2) and the results from (1) to show that if the parametrization $\mathbf{x}$ is isothermal, then

$$\mathbf{x}_{uu} + \mathbf{x}_{vv} = 2EH\mathbf{n}.$$

Corollary 16. A surface M with an isothermal parametrization $\mathbf{x}(u, v) = \big(x^1(u, v), x^2(u, v), x^3(u, v)\big)$ is minimal $\Longleftrightarrow$ x^1, x^2, x^3 are harmonic.

Proof. ($\Rightarrow$) If M is minimal, then $H = 0$ and so by Theorem 15 $\mathbf{x}_{uu} + \mathbf{x}_{vv} = 0$, and hence the coordinate functions are harmonic. ($\Leftarrow$) Suppose x^1, x^2, x^3 are harmonic. Then $\mathbf{x}_{uu} + \mathbf{x}_{vv} = 0$, and so by Theorem 15 $2(\mathbf{x}_u \cdot \mathbf{x}_u)H\mathbf{n} = 0$. But $\mathbf{n} \neq 0$ and $E = \mathbf{x}_u \cdot \mathbf{x}_u \neq 0$. Hence, $H = 0$ and M is minimal. □

Exercise 4. Let $\mathbf{x}$ and $\mathbf{y}$ be isothermal parametrizations of minimal surfaces such that their component functions are pairwise harmonic conjugates. In such a case, $\mathbf{x}$ and $\mathbf{y}$ are called *conjugate minimal surfaces.*

1. An isothermal parametrization for the catenoid and for the helicoid are, respectively,

$$\begin{aligned}\mathbf{x}_c(u,v) =&(a\cosh v\cos u, a\cosh v\sin u, av), \text{ and}\\ \mathbf{x}_h(u,v) =&(a\sinh v\cos u, a\sinh v\sin u, au).\end{aligned}$$

   Show that the helicoid and the catenoid are conjugate minimal surfaces.

2. Prove that given two conjugate minimal surfaces, $\mathbf{x}$ and $\mathbf{y}$, all surfaces of the one-parameter family

$$\mathbf{z} = (\cos t)\mathbf{x} + (\sin t)\mathbf{y} \tag{7}$$

   have the same fundamental form: $E =< \mathbf{x}_u, \mathbf{x}_u >=< \mathbf{y}_u, \mathbf{y}_u >$, $F = 0$, $G =< \mathbf{x}_v, \mathbf{x}_v >=< \mathbf{y}_v, \mathbf{y}_v >$.

3. Prove that all the surfaces in the one-parameter family (7) are minimal for all $t \in \mathbb{R}$.

Thus, any two conjugate minimal surfaces can be joined through a one-parameter family of minimal surfaces, and the first fundamental form of this family is independent of t.

---

3 Weierstrass Representation

The Weierstrass representation provides us a basic formula using functions from complex analysis for creating minimal surfaces.

Suppose M is a minimal surface with an isothermal parametrization $\mathbf{x}(u,v)$. Let $z = u + iv$. Formally, we can solve for u, v in terms of z, $\overline{z}$ to get $u = \frac{z+\overline{z}}{2}$ and $v = \frac{z-\overline{z}}{2i}$. Then the parametrization of M can be written as:

$$\mathbf{x}(z,\ \overline{z}) = \big(x^1(z,\ \overline{z}), x^2(z,\ \overline{z}), x^3(z,\ \overline{z})\big).$$

Exercise 5. The notion of analyticity requires that the function $f(x,y) = u(x,y) + iv(x,y)$ can be written in terms of $z = x + iy$ alone, without using $\overline{z} = x - iy$. Then we have the complex differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

Show that f is analytic $\Longleftrightarrow \frac{\partial f}{\partial \overline{z}} = 0$.

Exercise 6. Prove that

$$f_{uu} + f_{vv} = 4\left(\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial\, \overline{z}}\right)\right). \tag{8}$$

Theorem 17. Let M be a surface with parametrization $\mathbf{x} = (x^1, x^2, x^3)$ and let $\phi = (\varphi^1, \varphi^2, \varphi^3)$, where $\varphi^k = \frac{\partial x^k}{\partial z}$. Then $\mathbf{x}$ is isothermal $\Longleftrightarrow \phi^2 = 0$. If $\mathbf{x}$ is isothermal, then M is minimal $\Longleftrightarrow$ each φ^k is analytic.

Proof. Notice $(x_z^k)^2 = [\frac{1}{2}(x_u^k - ix_v^k)]^2 = \frac{1}{4}((x_u^k)^2 - (x_v^k)^2 - 2ix_u^k x_v^k)$. Therefore,

$$\begin{aligned}\phi^2 =&(\varphi^1)^2 + (\varphi^2)^2 + (\varphi^3)^2\\ =&\frac{1}{4}\left(\sum_{k=1}^{3}(x_u^k)^2 - \sum_{k=1}^{3}(x_v^k)^2 - 2i\sum_{k=1}^{3} x_u^k x_v^k\right)\\ =&\frac{1}{4}\left(|\mathbf{x}_u|^2 - |\mathbf{x}_v|^2 - 2i\mathbf{x}_u \cdot \mathbf{x}_v\right)\\ =&\frac{1}{4}(E - G - 2iF).\end{aligned}$$

Thus, $\mathbf{x}$ is isothermal $\Longleftrightarrow E = G, F = 0 \Longleftrightarrow \phi^2 = 0$.

Now suppose that $\mathbf{x}$ is isothermal. By Corollary 16, it suffices to show that for each k, x^k is harmonic $\Longleftrightarrow \varphi^k$ is analytic. Using eq (8) this follows because

$$x_{uu}^k + x_{vv}^k = 4\left(\frac{\partial}{\partial\, \overline{z}}\left(\frac{\partial x^k}{\partial z}\right)\right) = 4\left(\frac{\partial}{\partial\, \overline{z}}\left(\varphi^k\right)\right).$$

□

Note that if $\mathbf{x}$ is isothermnal

$$\begin{aligned}|\phi|^2 =&|x_z^1|^2 + |x_z^2|^2 + |x_z^3|^2\\ =&\frac{1}{4}\left(\sum_{k=1}^{3}(x_u^k)^2 + \sum_{k=1}^{3}(x_v^k)^2\right)\\ =&\frac{1}{4}\left(|\mathbf{x}_u|^2 + |\mathbf{x}_v|^2\right)\\ =&\frac{1}{4}(E+G)\\ =&\frac{E}{2}.\end{aligned}$$

So if $|\phi|^2 = 0$, then M degenerates to a point.

By Theorem 17, we can write

$$x^k(z\ \overline{z}) = c_k + 2\,\mathrm{Re}\int \varphi^k dz. \tag{9}$$

This is because

$$
\begin{aligned}
\varphi^k dz =& \frac{1}{2}[(x_u^k - ix_v^k)(du + idv)] = \frac{1}{2}[x_u^k du + x_v^k dv + i(x_u^k dv - x_v^k du)],\\
\overline{\varphi^k} dz =& \frac{1}{2}[(x_u^k + ix_v^k)(du - idv)] = \frac{1}{2}[x_u^k du + x_v^k dv - i(x_u^k dv - x_v^k du)],
\end{aligned}
$$

and so

$$
\begin{aligned}
dx^k =& \frac{\partial x^k}{\partial z} dz + \frac{\partial x^k}{\partial \overline{z}} d\,\overline{z}\\
=& \varphi^k dz + \overline{\varphi^k} d\,\overline{z}\\
=& \varphi^k dz + \overline{\varphi^k dz}\\
= 2\,\mathrm{Re}\,\varphi^k dz.
\end{aligned}
$$

Summary: We can find a formula for constructing minimal surfaces by determining analytic functions ϕ^k $(k = 1, 2, 3)$ such that

$$\phi^2 = 0 \qquad \text{and } |\phi|^2 \neq 0.$$

Consider

$$
\begin{aligned}
\varphi^1 =& p(1 + q^2)\\
\varphi^2 =& -ip(1 - q^2)\\
\varphi^3 =& -2ipq.
\end{aligned}
$$

Then

$$
\begin{aligned}
\phi^2 =& [p(1 + q^2)]^2 + [-ip(1 - q^2)]^2 + [-2ipq]^2\\
=& [p^2 + 2p^2q^2 + p^2q^4] - [p^2 - 2p^2q^2 + p^2q^4] - [4p^2q^2]\\
=& 0,
\end{aligned}
$$

and

$$
\begin{aligned}
|\phi|^2 =& |p(1 + q^2)|^2 + |-ip(1 - q^2)|^2 + |-2ipq|^2\\
=& |p|^2[(1 + q^2)(1 + \overline{q}^2) + (1 - q^2)(1 - \overline{q}^2) + 4q\,\overline{q}]\\
=& |p|^2[2(1 + 2q\,\overline{q} + q^2\,\overline{q}^2)\\
=& 4|p|^2(1 + |q|^2)|^2 \neq 0 \qquad \text{(note: if } p = 0\text{, then } \varphi^k = 0 \text{ for all } k).
\end{aligned}
$$

Theorem 18 (Weierstrass Representation I). Let p an analytic function and q a meromorphic function in some domain $\Omega \in \mathbb{C}$, having the property that at each point where q has a pole of order m, p has a zero of order at least

$2m$. Then every regular minimal surface has a local isothermal parametric representation of the form

$$\begin{aligned} X &= (x_1(z), x_2(z), x_3(z)) \\ &= \Big(\mathrm{Re}\Big\{ \int_0^z p(1+q^2)dw \Big\}, \\ &\qquad\qquad \mathrm{Re}\Big\{ \int_0^z -ip(1-q^2)dw \Big\}, \\ &\qquad\qquad\qquad \mathrm{Re}\Big\{ \int_0^z -2ipq dw \Big\} \Big). \end{aligned}$$

Example 19.

1. For $p = 1, q = iz$, we get

$$X = \Big(\mathrm{Re}\Big\{ z - \frac{1}{3}z^3 \Big\}, \mathrm{Re}\Big\{ -i\Big(z + \frac{1}{3}z^3\Big) \Big\}, \mathrm{Re}\Big\{ z^2 \Big\} \Big)$$

   which gives Enneper's surface.

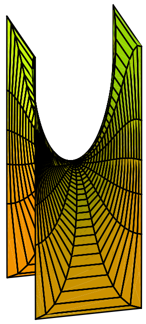
2. Using $p = 1/(1-z^4)$ and $q = iz$, the Weierstrass representation yields

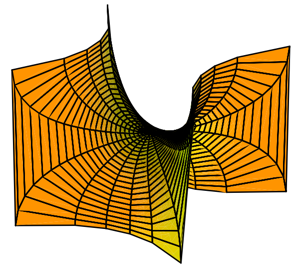
$$\begin{aligned} X = &\Bigg(\mathrm{Re}\left\{ \frac{i}{2} \log\left(\frac{z+i}{z-i} \right) \right\}, \\ &\mathrm{Re}\left\{ -\frac{i}{2} \log\left(\frac{1+z}{1-z} \right) \right\}, \\ &\mathrm{Re}\left\{ \frac{1}{2} \log\left(\frac{1+z^2}{1-z^2} \right) \right\} \Bigg) \end{aligned}$$

   and generates Scherk's doubly-periodic surface. Keeping p the same but changing q to $q = z$, we have

$$\begin{aligned} X = &\Bigg(\mathrm{Re}\left\{ \frac{1}{2} \log\left(\frac{z+i}{z-i} \right) \right\}, \\ &\mathrm{Re}\left\{ -\frac{1}{2} \log\left(\frac{1+z}{1-z} \right) \right\}, \\ &\mathrm{Re}\left\{ -\frac{i}{2} \log\left(\frac{1+z^2}{1-z^2} \right) \right\} \Bigg) \end{aligned}$$

   which forms Scherk's singly-periodic surface.





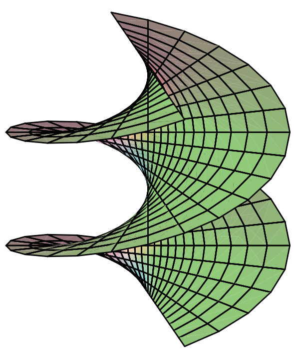
Scherk's doubly-periodic surface

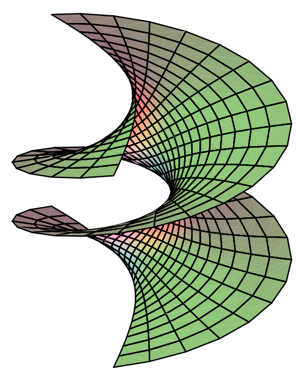
Scherk's singly-periodic surface

3. For $p = 1, q = \frac{e^{i\theta}}{z}$, we get

$$X = \left(\mathrm{Re}\left\{ z - \frac{e^{2i\theta}}{z} \right\}, \mathrm{Re}\left\{ -i\left(z + \frac{e^{2i\theta}}{z} \right) \right\}, \mathrm{Re}\left\{ -2ie^{i\theta} \log z \right\} \right)$$

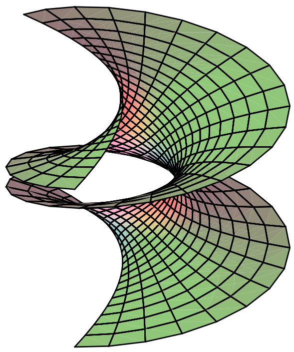
which represents a family of minimal surfaces that vary from the helicoid when $\theta = 0$ to the catenoid when $\theta = \pi/2$.



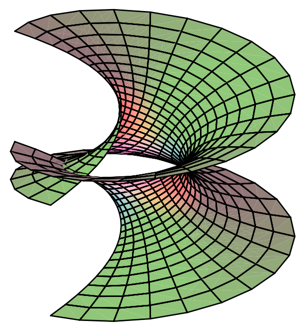


(a) Helicoid ($\theta = 0$)

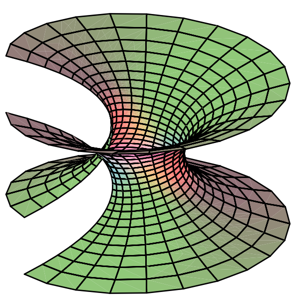
(b) Associated surface ($\theta = \frac{\pi}{6}$)



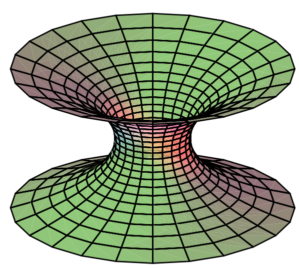
(c)Associated surface ($\theta = \frac{\pi}{4}$)



(d) Associated surface($\theta = \frac{\pi}{3}$)



(e) Associated surface ($\theta = \frac{5\pi}{12}$)



(f) Catenoid ($\theta = \frac{\pi}{2}$)

Another benefit of the Weierstrass representation is that we can investigate properties of minimal surfaces by just using the coordinate functions φ^k.

4 Minimal Surfaces and Harmonic Univalent Mappings

We can use other representations for $\phi = (\varphi^1, \varphi^2, \varphi^3)$ to form different Weierstrass representations as long as $\phi^2 = 0$ and $|\phi| \neq 0$. Since we are interested in planar harmonic mappings, a natural choice is to consider

$$\begin{aligned}
x^1 =& \operatorname{Re}(h+g) = \operatorname{Re}\int (h'+g')\,dz = 2\operatorname{Re}\int \varphi^1\,dz \\
x^2 =& \operatorname{Im}(h-g) = \operatorname{Re}\int -i(h'-g')\,dz = 2\operatorname{Re}\int \varphi^2\,dz \\
x^3 =& 2\operatorname{Re}\int \varphi^3\,dz
\end{aligned}$$

and then solve for φ^3.

$$\begin{aligned}
0 = \phi^2 =& \frac{1}{2}\left\{[h'+g']^2 + [-i(h'-g')]^2 + [\varphi^3]^2\right\} \\
=& \frac{1}{2}\left\{(h')^2 + 2h'g' + (g')^2 - (h')^2 + 2h'g' - (g')^2 + (\varphi^3)^2\right\} \\
\Rightarrow \varphi^3 =& \frac{1}{2}\sqrt{-4h'g'} = ih'\sqrt{g'/h'}
\end{aligned}$$

We need φ^3 to be analytic and so we require the dilatation $\omega = g'/h'$ to be a perfect square.

Theorem 20 (Weierstrass Representation II). If a minimal graph $\{(u, v, F(u, v)) : u + iv \in \Omega\}$ is parametrized by sense-preserving isothermal parameters $z = x + iy \in \mathbb{D}$, the projection onto its base plane defines a harmonic mapping $w = u + iv = f(z)$ of $\mathbb{D}$ onto Ω whose dilatation is the square of an analytic function. Conversely, if $f = h + \overline{g}$ is a sense-preserving harmonic mapping of $\mathbb{D}$ onto some domain Ω with dilatation $\omega = q^2$ for some function q analytic in $\mathbb{D}$, then the formulas

$$\begin{aligned}
u &= \operatorname{Re}\{h(z) + g(z)\}, \\
v &= \operatorname{Im}\{h(z) - g(z)\}, \\
F(u, v) &= 2\operatorname{Im}\left\{\int_0^z \sqrt{g'(\zeta)h'(\zeta)}d\zeta\right\}
\end{aligned}$$

define by isothermal parameters a minimal graph whose projection is f.

Example 21. Polynomial mapping and Enneper's surface

Recall the polynomial harmonic univalent mappings

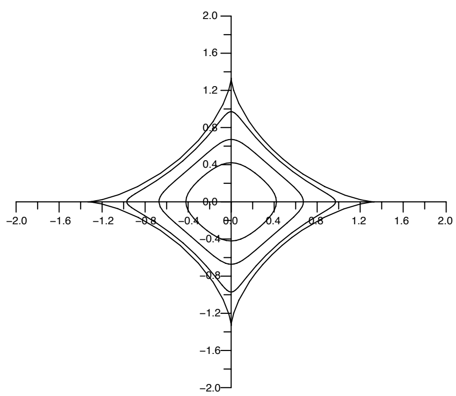
$$\begin{aligned}
f(z) =& z + \frac{1}{3}\overline{z}^3 \\
=& \operatorname{Re}\left(z + \frac{1}{3}z^3\right) \\
=& \operatorname{Im}\left(z - \frac{1}{3}z^3\right).
\end{aligned}$$

Hence, $h(z) = z$ and $g(z) = 1/3z^3$ and

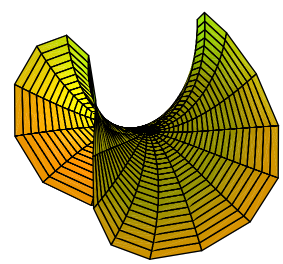
$$F(u,v) = 2\,\mathrm{Im}\int_0^z \sqrt{\zeta^2}\,d\zeta = \mathrm{Im}\,z^2.$$

This yields a parametrization of Enneper's minimal surface:

$$X = \left(\mathrm{Re}\left\{z + \frac{1}{3}z^3\right\}, \mathrm{Im}\left\{z - \frac{1}{3}z^3\right\}, \mathrm{Im}\left\{z^2\right\}\right)$$



Projection of
Enneper's Surface



Enneper's
Minimal Surface

Theorem 20 allows to create minimal surfaces from planar harmonic mappings. One way to do this is to shear an analytic univalent function that is convex in the horizontal direction and require that the dilatation is the square of an analytic function.

Example 22. Let $h(z) - g(z) = \dfrac{1}{2}\log\left(\dfrac{1+z}{1-z}\right)$ and $\omega = g'(z)/h'(z) = m^2z^2$, where $|m| \leq 1$. Solving for h and g yields

$$\begin{aligned} h(z) &= \frac{1}{2(1-m^2)}\log\left(\frac{1+z}{1-z}\right) + \frac{m}{2(m^2-1)}\log\left(\frac{1+mz}{1-mz}\right) \\ g(z) &= \frac{m^2}{2(1-m^2)}\log\left(\frac{1+z}{1-z}\right) + \frac{m}{2(m^2-1)}\log\left(\frac{1+mz}{1-mz}\right). \end{aligned}$$

For every m such that $|m| = 1$, the image of $\mathbb{D}$ under $f = h + \overline{g}$ is a parallelogram. Now consider the minimal surfaces constructed from this shearing. Note that

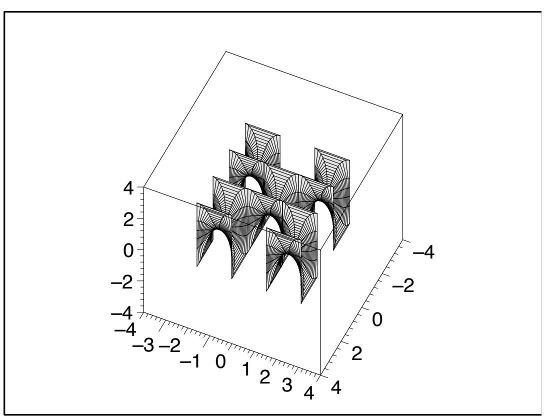
$$F(u,v) = \mathrm{Im}\left\{ \frac{m}{1-m^2} \log\left(\frac{1-m^2z^2}{1-z^2} \right) \right\}.$$

This forms a one-parameter family of slanted Scherk surfaces that range from the canonical Scherk doubly periodic surface to the helicoid. In particular, for $m = i$ the resulting minimal surface is Scherk's doubly periodic surface. With $m = e^{i\theta}$ and letting θ decreases between $\pi/2$ and 0, we get sheared transformations of Scherk's doubly periodic surface. In the limit $\theta = 0$ we derive

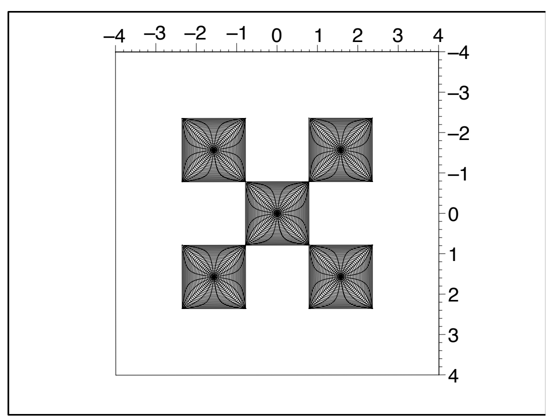
$$X = (x_1, x_2, x_3) = \left(\frac{1}{2}\sinh u \cos v, \frac{1}{2}v, \frac{1}{2}\sinh u \sin v \right),$$

which is an equation of a helicoid.

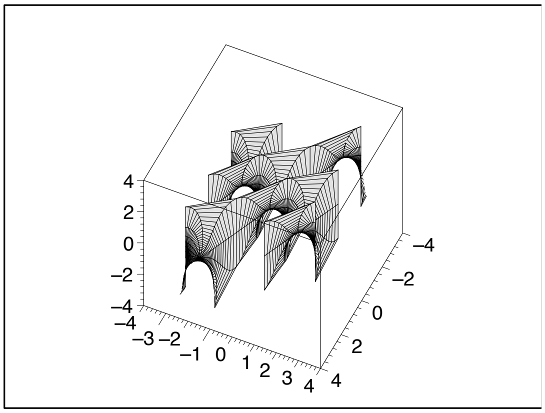
he Schwarz reflection principle allows one to create larger periodic minimal surfaces by reflecting a known surface through lines and planes of symmetry. Because the projection of these minimal surfaces is a parallelogram, we can apply the Schwarz reflection principle to combine one of these minimal surfaces with copies of itself to form a checkerboard like conglomeration of minimal surfaces.



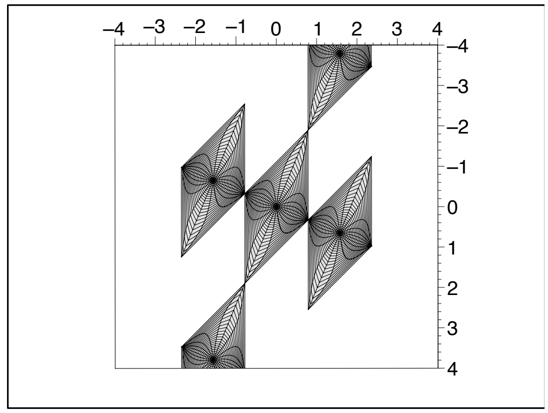
(a) Five Pieces of Scherk's Surface ($m = e^{i\pi/2}$)



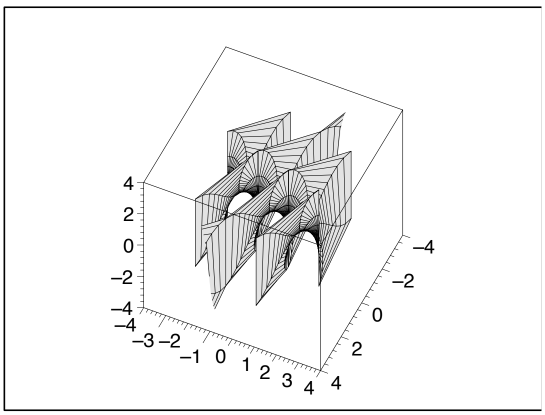
(b) Projection of (a) onto $\mathbb{C}-$plane



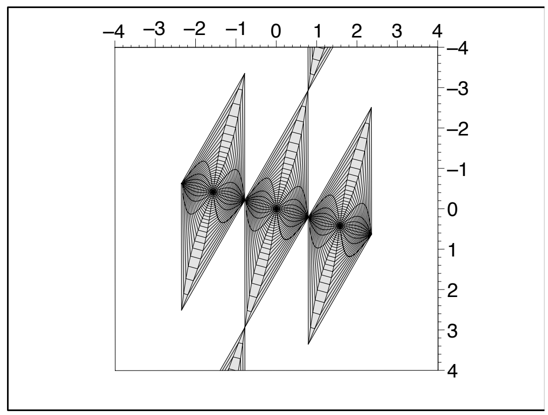
(c) Five Pieces of Scherk's Slanted Surface ($m = e^{i\pi/4}$)



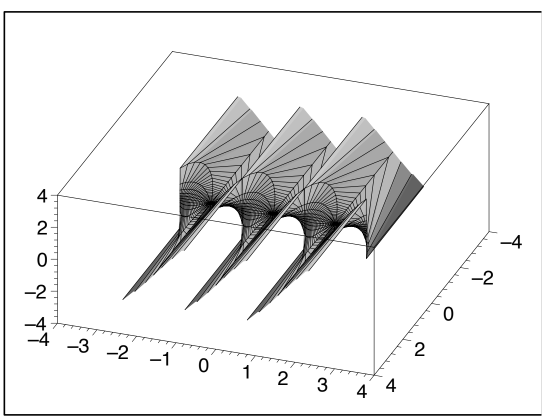
(d) Projection of (c) onto $\mathbb{C}-$plane



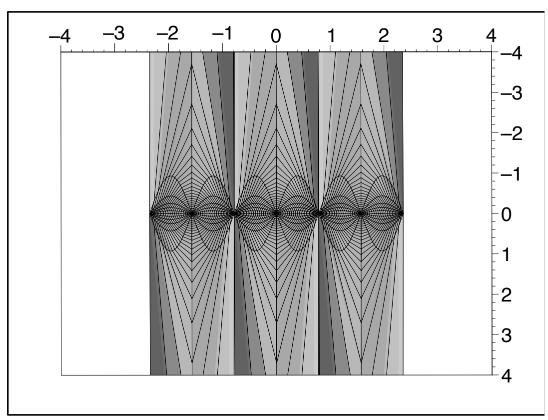
(e) Five Pieces of Scherk's Slanted Surface ($m = e^{i\pi/6}$)



(f) Projection of (e) onto $\mathbb{C}-$plane



(g) Three Pieces of the Helicoid ($m = 1$)



(h) Projection of (g) onto $\mathbb{C}-$plane

Problems to Investigate 1.

1. Find known minimal surfaces that relate to planar harmonic mappings that have dilatations that are perfect squares. Here are some examples:

   (a) the polynomial map given by

   $$f(z) = z + \frac{1}{3}\overline{z}^3$$

   is the projection of Enneper's surface. What minimal surfaces correspond to other polynomial harmonic maps?

   (b) the square map given by

   $$f(z) = f(z) + \overline{g(z)} = \operatorname{Re}\left[\frac{i}{2}\log\left(\frac{1-iz}{1+iz}\right)\right] + i\operatorname{Im}\left[\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right]$$

   is the projection of Scherk doubly-periodic surface. Generally, any convex $2n$-polygonal image in $\mathbb{C}$ is the projection of a Jenkins-Serre minimal surface.

   (c) the horizontal strip map

   $$f_1(z) = h_1(z) + \overline{g_1(z)} = \operatorname{Re}\left[\frac{z}{1-z^2}\right] + i\operatorname{Im}\left[\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right]$$

   is the helicoid;

   (d) the map

   $$f_2(z) = h_2(z) + \overline{g_2(z)} = \operatorname{Re}\left[\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right] + i\operatorname{Im}\left[\frac{z}{1-z^2}\right]$$

   is the catenoid.

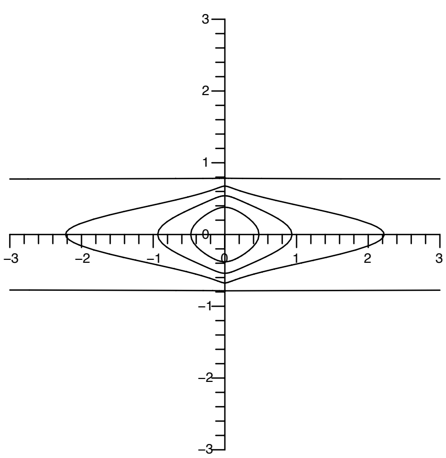
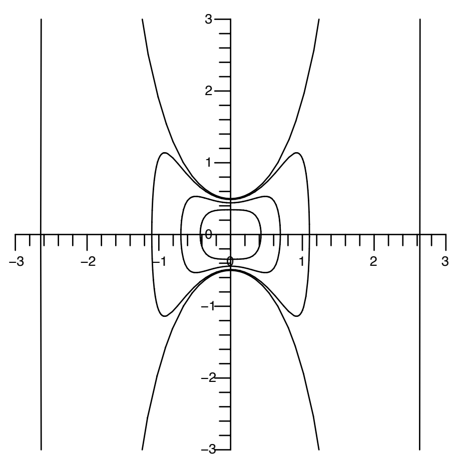


Image of $\mathbb{D}$ under f_1



(h) Image of $\mathbb{D}$ under f_2

Problem: What minimal surface do we get when we shear $z/(1-z)^2$ using $g'(z) = z^2 h'(z)$?

2. Given a specific minimal surface, find the corresponding planar harmonic mappings. For example, determine $f = h + \overline{g}$ that corresponds with the twisted Scherk singly-periodic surface.

3. Recall that a minimal surface is defined as a surface M for which $H = \frac{k_1+k_2}{2} = 0$ at each point $p \in M$, where k_1, k_2 are the maximum and minimum normal curvatures at p. Another curvature on a surface is its Gaussian curvature defined by

$$K = k_1 k_2.$$

Like the mean curvature, the Gaussian curvature can be written in terms of the coefficients of the first and second fundamental forms. In particular,

$$K = \frac{eg - f^2}{EG - F^2}. \tag{10}$$

In the case we have an isothermal parametrization of a surface M, eq (10) reduces to

$$K = -\frac{1}{\lambda^2}\Delta(\log \lambda),$$

where $\lambda^2 = E$ and $\Delta = \partial^2/\partial u^2 + \partial^2/\partial v^2$ is the Laplacian. This means that the Gaussian curvature depends only on the coefficients of the first fundamental form, and hence K does not change as a surface is deformed as long as the change does not stretch the surface. This is known as Gauss' *theorema egregium* (i.e., "beautiful theorem").

It can be shown that if M is a minimal surface, then

$$K = -\frac{4|q'|^2}{|p|^2(1+|q|^2)^4} = -\frac{|\omega'|^2}{|h'g'|(1+|\omega|)^4},$$

where p, q are from the Weierstrass Representation I and h, g, $\omega = g'/h'$ are from Weierstrass Representation II (and the corresponding planar harmonic mapping). This allows us to estimate $|K|$ by using function theory. In particular, by the Schwarz-Pick lemma

$$\frac{|q'(z)|}{1-|q(z)|^2} \le \frac{1}{1-|z|^2}, \qquad \forall z \in \mathbb{D}.$$

Thus, at the point of the surface above $f(0)$ we have

$$\begin{aligned}
|K| \le & \frac{4(1-|q(0)|^2)^2}{|p(0)|^2(1+|q(0)|^2)^4} = \frac{4(1-|\omega(0)|)^2}{(|h'(0)|+|g'(0)|)^2(1+|\omega(0)|)^2} \\
\le & \frac{4}{(|h'(0)|+|g'(0)|)^2} \le \frac{4}{|h'(0)|^2+|g'(0)|^2}.
\end{aligned}$$

Example 23. Assume that M is a minimal graph over a domain Ω and that $h(0) = 0 = g(0)$.

(a) in the case the minimal surface is over $\mathbb{D}$, Heinz' lemma proves that $|h'(0)|^2 + |g'(0)|^2 \geq 27/(4\pi^2)$. Hence, the Gaussian curvature at the point over the origin satisfies

$$|K| \leq \frac{16\pi^2}{27} \approx 5.85$$

However, this result is not sharp, because Heinz' result has equality for the planar harmonic map that sends $\mathbb{D}$ onto the regular triangle, and this map has $\omega(z) = z$ which does not lift to a minimal surface since ω is not a perfect square. Since the harmonic square mapping has dilatation z^2, we have the following conjecture

Conjecture 1. If M is a minimal graph over $\mathbb{D}$, then the Gaussian curvature at the point over the origin satisfies

$$|K| \leq \frac{\pi^2}{2} \approx 4.93$$

(b) In the case the minimal surface is over the infinite horizontal strip $\Omega = \{z \in \mathbb{C} \, \big| \, |\operatorname{Im} z| < \frac{\pi}{4}\}$, then Hengartner and Schober showed that the Gaussian curvature at the point over $\alpha = a + ib \in \Omega$ satisfies the sharp inequalities

$$|K| \begin{cases} \leq 4, & \text{if } b = 0, \\ < 4\sec^2(2b), & \text{if } b \neq 0. \end{cases}$$

Problem: Using this approach, find estimates on the bound of K at points for other minimal surfaces.

4. Krust Theorem in minimal surface theory states that if M is a minimal graph over a convex domain, then all of the associated surfaces of M are also minimal graphs. Using a convolution theorem by Clunie and Sheil-Small, it is easy to show that these associated surfaces are over close-to-convex domains. This suggests another avenue for further study–use theorems from one of these fields about forming or combining maps to get further results in the other field.